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arXiv:hep-th/9907167 v2 22 Dec 1999

Short-distance analysis for algebraic euclidean field theory

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January 29, 2002

Abstract

Recently D. Buchholz and R. Verch have proposed a method for implementing in algebraic quantum field theory ideas from renormalization group analysis of short-distance (high energy) behavior by passing to certain scaling limit theories. Buchholz and Verch distinguish between different types of theories where the limit is unique, degenerate, or classical, and the method allows in principle to extract the 'ultra-particle' content of a given model, i.e. to identify particles (like quarks and gluons) that are not visible at finite distances due to 'confinement'. It is therefore of great importance for the physical interpretation of the theory. The method has been illustrated in a simple model in with some rather surprising results.

This paper will focus on the question how the short distance behavior of models defined by euclidean means is reflected in the corresponding behavior of their Minkowski counterparts. More specifically, we shall prove that if a euclidean theory has some short distance limit, then it is possible to pass from this limit theory to a theory on Minkowski space, which is a short distance limit of the Minkowski space theory corresponding to the original euclidean theory.

1 Introduction

In the past three decades several approaches have been developed that incorporate the physical principles of quantum field theory (QFTh) in a mathematically rigorous fashion. Among these rigorous approaches the framework of algebraic quantum field theory is probably the most highly developed. Its basic objects are algebras of observables, indexed by domains of space-time where the observables can be measured. Hence the interpretation of the basic structure is rather clear-cut. Many deep, model independent results have been obtained within this framework, which has also proved well adapted to the construction and analysis of models in two dimensional conformal field theory.

On the other hand, models of quantum fields based on concrete lagrangians are usually constructed by means of euclidean functional integrals [16]. From the constructive point of view these methods have many advantages, for the basic algebraic structure is commutative and powerful tools from classical statistical mechanics can be applied. The passage from such objects in euclidean space to algebraic quantum field theory on Minkowski space time is mathematically a highly non-trivial operation, however, and it is by no means clear in general how properties of the former are reflected in latter. Hence the physical interpretation is much less obvious than in algebraic quantum field theory. When the euclidean theory has only been proved to exist in a bounded euclidean domain, as is the case for the four dimensional Yang-Mills model constructed by Magnen, Rivasseau and Senéor [19], the passage itself is an important unsolved problem.

Recently D. Buchholz and R. Verch have proposed a method for implementing in algebraic quantum field theory ideas from renormalization group analysis of short-distance (high energy) behavior by passing to certain scaling limit theories [9, 8, 7, 6, 5, 4]. Buchholz and Verch distinguish between different types of theories where the limit is unique, degenerate, or classical, and the method allows in principle to extract the 'ultraparticle' content of a given model, i.e. to identify particles (like quarks and gluons) that are not visible at finite distances due to 'confinement'. It is therefore of great importance for the physical interpretation of the theory. The method has been illustrated in a simple model in [5] with some rather surprising results.

This paper will focus on the question how the short distance behavior of models defined by euclidean means is reflected in the corresponding behavior of their Minkowski counterparts. More specifically, we shall prove that if a euclidean theory has some short distance limit, then it is possible to pass from this limit theory to a theory on Minowski space, which is a short distance limit of the Minkowski space theory corresponding to the original euclidean theory.

The present status of algebraic euclidean field theory. The techniques of euclidean field theory (EFTh) [16] have proved to be very powerful for the construction of interacting quantum field theory models and often superior to the method of canonical quantization in Minkowski space. For instance, existence of the ϕ_3^4 model as a Wightman quantum field theory has been established by using euclidean methods [11, 25, 20] combined with the Osterwalder-Schrader reconstruction theorem [21]. Within a hamiltonian framework essentially only the proof of the positivity of the energy has been carried out. Also in cases where a direct Minkowski space construction is possible, as in the $P(\phi)_2$ and Yukawa$_2$ models, euclidean techniques may simplify things considerably, e.g. in in the proof of of Poincaré covariance or discussions of phase transitions and symmetry breaking. In these constructions the key objects are usually the euclidean Greens functions, or Schwinger distributions, $\mathfrak{S}_n$ that are represented as moments of a measure $d\mu$ on a space of distributions $S'(\mathbb{R}^d)$

$$\mathfrak{S}_n(x_1, \cdots, x_n) = \int \mathrm{d}\mu(\phi)\ \phi(x_1)\cdots\phi(x_n).$$

Methods from statistical mechanics like renormalization group analysis [14] and cluster expansions [3] can be applied in order to perform the continuum and the infinite volume limits of lattice regularized models. But the construction of Schwinger distributions is not enough, the problem of linking them to physics in Minkowski space has to be addressed.

The Osterwalder-Schrader reconstruction theorems [21] connect the Schwinger distributions of an euclidean field theory with the Wightman distributions of a quantum field theory on Minkowski space. Powerful as these theorems are, there are several reasons why they can not be considered as the final answer to the problem of linking euclidean and Minkowski space theories. For one thing, the conditions of the equivalence theorem (Theorem $E \leftrightarrow R$ in [21]) are extremely hard to verify, while the convenient sufficient conditions for the passage from Schwinger distributions to Wightman distributions (Theorem E'(or E'')$\rightarrow R'$ in [21]) are most probably too restrictive in general. Secondly, and this is a more important point than the first for the present discussion, the results in [21] do not allow one to conclude that a local net of observable algebras in the sense of algebraic quantum field theory can be obtained from the Schwinger distributions. In fact, the question when a Wightman quantum field gives rise to such a net is quite a delicate one, see [1] for a review. Useful sufficient conditions are known, however [16, 10]. In the models with point fields constructed so far these conditions are fulfilled, but it appears very unlikely that they will be so in general.

The third point is that Schwinger distributions, which are euclidean expectation values of point fields, may not be adequate in gauge theories and one should rather consider expectation values of extended objects localized around loops or even strings extending to infinity. An Osterwalder-Schrader-type reconstruction scheme which can be applied to correlation functions of loops and strings was established by E. Seiler [24] and J. Fröhlich, K. Osterwalder and E. Seiler [13]. Local commutativity of the reconstructed observables remained an open problem in this work, however.

A C*-algebraic version of the reconstruction theorems in [21, 10, 16], which to a certain extent generalizes the previous considerations in [13, 24] and solves the locality problem, has been worked out in [22]. The starting point of this analysis is a net $\mathfrak{B}$ of C*-algebras, indexed by regions in euclidean space, and acted upon by an action γ of the euclidean group by automorphisms of $\mathfrak{E}$. The third ingredient is a continuous euclidean invariant and reflexion positive [16] functional η on $\mathfrak{B}$. It is shown in [22] that for a given euclidean field $(\mathfrak{B}, \beta, \eta)$ a Haag-Kastler net $\mathfrak{A}$ on Minkowski space, a covariant action of the Poincaré group by automorphisms α on $\mathfrak{A}$ as well as a vacuum state ω on $\mathfrak{A}$ can directly be reconstructed from $(\mathfrak{B}, \beta, \eta)$. The main advantage of the C*-algebraic framework, is that one only deals with bounded operators from the outset. This is also important for the proof of locality for the constructed Haag-Kastler net. Finally, we mention that there are indications that a C*-algebraic point of view also enlarges the variety of constructible euclidean field theory models [23].

Scaling algebras and renormalization group. A general approach for the analysis of the high energy properties of a given quantum field theory model has been developed by D. Buchholz and R. Verch [9, 8, 7, 6, 5, 4]. The starting point of their analysis is a quantum field theory formulated within the C*-algberaic approach as it has been introduced by R. Haag and D. Kastler [17, 18]. We briefly recall here the mathematical description of this framework.

A $\mathrm{P}_+^\uparrow$-covariant Haag-Kastler net is an inclusion preserving prescription which assigns to each double cone $\mathscr{O} = V_+ + x \cap V_- + y$ a unital C*-subalgebra $\mathfrak{A}(\mathscr{O}) \subset \mathfrak{A}$ of a C*-algebra $\mathfrak{A}$. [1] The self-adjoint elements within a local algebra $\mathfrak{A}(\mathscr{O})$ correspond to observables which can be measured within the spacetime region $\mathscr{O}$. The Poincaré group $\mathrm{P}_+^\uparrow$ acts covariantly on the net $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$, i.e. there is a group homomorphism α from the Poincaré group into the automorphism group of $\mathfrak{A}$, such that $\alpha_g \mathfrak{A}(\mathscr{O}) = \mathfrak{A}(g\mathscr{O})$ for each Poincaré transformation g. The concept of locality is encoded by the property that, if $\mathscr{O}, \mathscr{O}_1$ are two spacelike separated regions, then the operators in $\mathfrak{A}(\mathscr{O})$ com-

[1] Here $V_\pm = \{x | x^2 > 0, \pm x^0 > 0\}$ is the forward (backward) light cone in Minkowski space.

mute with those in $\mathfrak{A}(\mathscr{O}_1)$, i.e. two measurements which are performed in spacelike separated regions are commensurable. We write for the corresponding $\mathrm{P}_+^\uparrow$-covariant Haag-Kastler net $(\mathfrak{A}, \alpha)$.

One selection criterion for physical states, which is related to a stability requirement, is the so called *spectrum condition*. A state ω, ehich is subject to this condition, is called a *positive energy state*, characterized by the property that there exists a unitary strongly continuous representation U of the Poincaré group on the GNS Hilbert space of ω, implementing the automorphisms α_g in the GNS representation of ω, such that the spectrum of the generator of the translation group $x \mapsto U(x)$ is contained in the closed forward light cone $\bar{V}_+$. A particular class of positive energy states are the *vacuum states*, which also have the property to be *Poincaré invariant*, $\omega \circ \alpha_g = \omega$. This reflects the fact that within a vacuum there is no matter configuration which can distinguish a certain region in spacetime.

A triple $(\mathfrak{A}, \alpha, \omega)$, where $(\mathfrak{A}, \alpha)$ is a Haag-Kastler net and ω is a positive energy state is called a *quantum field*.

The concept of scaling algebra allows to express some basic ideas of renormalization group analysis within the algebraic framework. For a positive number $\lambda > 0$ one builds a new Haag-Kastler net $\mathscr{O} \mapsto \mathfrak{A}_\lambda(\mathscr{O})$ by defining the scaled algebra of a domain $\mathscr{O}$ by $\mathfrak{A}_\lambda(\mathscr{O}) := \mathfrak{A}(\lambda\mathscr{O})$ and putting $\alpha_{(\lambda,g)} := \alpha_{\lambda \circ g \circ \lambda^{-1}}$ for a Poincaré transformation g. Thus one keeps Minkowski space fixed and one interprets the properties of the given theory at small scales (high energy behavior) in terms of the modified theories $\mathscr{O} \mapsto \mathfrak{A}_\lambda(\mathscr{O})$.

The *scaling algebra* $\underline{\mathfrak{A}}$ is the C*-algebra which is generated by a certain class of bounded functions $\mathbf{a} : \mathbb{R}_+ \to \mathfrak{A}$ (see [5] and related work). The functions in $\underline{\mathfrak{A}}$ are regarded as *orbits* under the action of the renormalization group transformations which identify operators at scale 1 with operators at scale λ. This requires a particular scaling behavior in configuration space, namely an operator $\mathbf{a} \in \underline{\mathfrak{A}}$ is localized in $\mathscr{O}$ if for each $\lambda \in \mathbb{R}_+$ the operator $\mathbf{a}(\lambda)$ is localized in the scaled region $\lambda\mathscr{O}$. On the other hand, a condition for the scaling behavior in momentum space is needed in order to fix Planck's constant $\hbar$. Formulated in terms of the scaling algebra, the correct scaling in momentum space can be archieved by requiering that the continuety property

$$\lim_{g \to 1} \sup_{\lambda \in \mathbb{R}_+} \|\mathbf{a}(\lambda) - \alpha_{\lambda \circ g \circ \lambda^{-1}} \mathbf{a}(\lambda)\| \quad = \quad 0$$

is fulfilled for each $\mathbf{a} \in \underline{\mathfrak{A}}$. In other words, the group homomorphism $\underline{\alpha}$ from the Poincaré group into the automorphism group of $\underline{\mathfrak{A}}$, which is given by

$$(\underline{\alpha}_g \mathbf{a})(\lambda) \quad := \quad \alpha_{\lambda \circ g \circ \lambda^{-1}} \mathbf{a}(\lambda)$$

is strongly continuous.

Each physical state ω (in particular a vacuum state) of the underlying theory $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$ can be lifted to a physical state $\underline{\omega}$ on the scaling algebra by the prescription

$$\langle \underline{\omega}, \mathbf{a} \rangle \quad := \quad \langle \omega, \mathbf{a}(1) \rangle \quad .$$

Hence the state $\underline{\omega}$ evaluates the renormalization group orbit $\mathbf{a}$ at scale $\lambda = 1$.

The group of scaling transformations σ_λ acts by automorphism on the scaling algebra in a natural fashion

$$(\sigma_\lambda \mathbf{a})(\lambda_1) \quad := \quad \mathbf{a}(\lambda_1 \lambda) \quad ,$$

leading to a net of states

$$\{\underline{\omega}_\lambda = \underline{\omega} \circ \sigma_\lambda | 0 < \lambda\}$$

which has, according to the weak-compactness of the set of states of a C*-algebra, weak limit points $\underline{\omega}_\zeta$ for $\lambda \to 0$ (the symbol ζ labels such a limit point). It has been proven in [8, 5] that all weak limit points $\underline{\omega}_\zeta$ are vacuum states and for each of them one obtains a quantum field $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$, where the algebra $\underline{\mathfrak{A}}_\zeta$ is defined by

$$\underline{\mathfrak{A}}_\zeta \quad := \quad \underline{\mathfrak{A}} / \underline{\pi}_\zeta^{-1}(0)$$

where $\underline{\pi}_\zeta$ is the GNS representation of $\underline{\omega}_\zeta$. The group homomorphism $\underline{\alpha}_\zeta$ is the lifting of $\underline{\alpha}$ to $\underline{\mathfrak{A}}_\zeta$, which exists, since the ideal $\underline{\pi}_\zeta^{-1}(0)$ is Poincaré invariant. These quantum fields are called *scaling limits* of the quantum field $(\mathfrak{A}, \alpha, \omega)$ and they describe the high energy behavior of the underlying theory. D. Buchholz and R. Verch distinguish three cases in order to classify the scaling limits:

(1) All scaling limit theories are equivalent, i.e. the scaling limit is unique.

(2) All scaling limit theories are just multiples of the identity, i.e. one obtains a classical scaling limit.

(3) Neither case *(1)* nor case *(2)* are valid, i.e. the scaling limit is degenerate.

Taking scaling limits and passing form EFTh to QFTh. For or a given euclidean field $(\mathfrak{B}, \beta, \eta)$ the short-distance behavior can be obtained by first passing by means of the construction procedure [22], which we are going to explain in Section 2 in more detail, to the corresponding quantum field theory model $(\mathfrak{A}, \alpha, \omega)$ and in a second step applying the analysis of D. Buchholz and R. Verch [8, 5] in order to get the scaling limit theories $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$. We illustrate this procedure digramatically in the following way:

$$(\mathfrak{B}, \beta, \eta) \xrightarrow{\text{recon}} (\mathfrak{A}, \alpha, \omega) \xrightarrow{\text{sclim}} (\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$$

The two step procedure, described above, is rather cumbersome and it is advantageous to be able to study the scaling limit theories directly on the euclidean level. More precisely, one wishes to build from a euclidean field $(\mathfrak{B}, \beta, \eta)$ the *euclidean scaling limit theory* $(\underline{\mathfrak{B}}_\zeta, \underline{\beta}_\zeta, \underline{\eta}_\zeta)$ first and then, in a second step, one constructs the corresponding Minkowski quantum field theory model which we denote by $(\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$, without underlining the the symbols.

$$(\mathfrak{B}, \beta, \eta) \xrightarrow{\text{sclim}} (\underline{\mathfrak{B}}_\zeta, \underline{\beta}_\zeta, \underline{\eta}_\zeta) \xrightarrow{\text{recon}} (\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$$

In Section 3 we describe how to build the euclidean scaling limit theory $(\underline{\mathfrak{B}}_\zeta, \underline{\beta}_\zeta, \underline{\eta}_\zeta)$ form a given euclidean field and we present there the main result of this paper, namely:

Theorem: *The quantum field theories, which can be reconstructed from euclidean scaling limit theories, are equivalent to scaling limit theories of the quantum field theory which can be reconstructed from the underlying euclidean field theory.*

Formally expressed in terms of diagrams, this means that the digram, given below, commutes in the sense of equivalence classes of quantum fields:

$$\begin{tikzcd}
(\mathfrak{B}, \beta, \eta) \arrow[r, "\text{sclim}"] \arrow[d, "\text{recon}"'] & (\underline{\mathfrak{B}}_\zeta, \underline{\beta}_\zeta, \underline{\eta}_\zeta) \arrow[d, "\text{recon}"] \\
(\mathfrak{A}, \alpha, \omega) \arrow[r, "\text{sclim}"] & (\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta) \cong (\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)
\end{tikzcd}$$

At this point, we briefly explain here, what equivalence of quantum fields means within our framework: Two quantum fields $(\mathfrak{A}, \alpha, \omega)$ and $(\hat{\mathfrak{A}}, \hat{\alpha}, \hat{\omega})$ are called equivalent if there exists an algebra isomorphism $\iota : \mathfrak{A} \to \hat{\mathfrak{A}}$ such that ι intertwines the group homomorphisms α and $\hat{\alpha}$, i.e. $\iota \circ \alpha_g = \hat{\alpha}_g \circ \iota$ holds true for each Poincaré transformation g, the states ω and $\hat{\omega}$ are related by $\hat{\omega} \circ \iota = \omega$, and the isomorphism ι respects the net structure, i.e. for each bounded and convex region $\mathscr{U} \subset \mathbb{R}^d$ the identity $\iota(\mathfrak{A}(\mathscr{U})) = \hat{\mathfrak{A}}(\mathscr{U})$ is valid.

2 From euclidean field theory to quantum field theory

Within this section we briefly discuss the ideas and strategies which have been developed in [22]. The starting point in the framework of algebraic euclidean field theory is an isotonous net

$$\mathscr{U} \longmapsto \mathfrak{B}(\mathscr{U}) \subset \mathfrak{B}$$

of C*-algebras, indexed by the set $\mathscr{K}^d$ of bounded convex regions $\mathscr{U}$ in $\mathbb{R}^d$, on which the euclidean group $\mathrm{E}(d)$ acts covariantly by automorphisms, i.e. there exists a group homomorphism β from the euclidean group $\mathrm{E}(d)$ into the automorphism group of $\mathfrak{B}$ such that

$$\beta_g \mathfrak{B}(\mathscr{U}) \quad = \quad \mathfrak{B}(g\mathscr{U})$$

for each $\mathscr{U} \in \mathscr{K}^d$ and for each $g \in \mathrm{E}(d)$. In order to implement the concept of locality within the euclidean framework, we assume that two operators commute if they are localized in disjoint regions, i.e. if $\mathscr{U}_1 \cap \mathscr{U}_2 = \emptyset$, then

$$[\mathfrak{B}(\mathscr{U}_1), \mathfrak{B}(\mathscr{U}_2)] \quad = \quad \{0\} \ .$$

A further ingredient for building quantum field theory models from euclidean data are *reflexion positive euclidean invariant regular states*. These states fulfill the following conditions:

Euclidean invariance: For each euclidean transformation $h \in \mathrm{E}(d)$: $\eta \circ \beta_h = \eta$.

Reflexion positivity: Let $e \in S^{d-1}$ be a euclidean time direction, then we denote by $\mathfrak{B}(e)$ the C*-algebra generated by operators which are localized in the half space $\mathbb{R}_+ e + e^\perp$, where $e^\perp$ is the hyperplane, orthogonal to e. A state η is reflexion positive if the sesquilinear form

$$\mathfrak{B}(e) \otimes \mathfrak{B}(e) \ni b_1 \otimes b_2 \mapsto \langle \eta, j_e(b_1) b_2 \rangle \tag{1}$$

is positive semidefinite. Here j_e is the antilinear involution

$$j_e(b) \quad = \quad \beta_{\theta_e}(b^*)$$

with $\theta_e : x \mapsto -2ex + x$.

Regularity: For each $b_1, b_2, b_3 \in \mathfrak{B}$ the map

$$h \mapsto \langle \eta, b_1 \beta_h(b_2) b_3 \rangle$$

is continuous.

Remarks.

(1) Without loss of generality, we may assume that the GNS representation π_η of η is faithful. Otherwise, we simply replace the algebra $\mathfrak{B}$ by the quotient $\mathfrak{B}/\pi_\eta^{-1}(0)$. Note that the automorphism β_h can be lifted to an automorphism on $\mathfrak{B}/\pi_\eta^{-1}(0)$ since η euclidean invariant.

(2) We claim here, that regularity is automatically fulfilled for η if the group homomorphism β is strongly continuous, i.e. $\lim_{g\to 1}\|\beta_g(b)-b\| = 0$ for each $b \in \mathfrak{B}$.

We showed in [22] how to construct from a given euclidean field a quantum field theory in a particular vacuum representation. In order to point out the relation between the euclidean field $(\mathfrak{B},\beta,\eta)$ and the minkowskian world, we briefly describe the construction of a Hilbert space $\mathcal{H}$ on which the physical observables are represented, the construction of a unitary strongly continuous representation of the Poincaré group on $\mathcal{H}$, as well as the Haag-Kastler net of local algebras.

Step 1: By dividing the null-space of the positive semidefinite sesquilinear form, introduced by Equation 1, and by taking the closure, we obtain a Hilbert space $\mathcal{H}$. The corresponding canonical projection onto the quotient is denoted by

$$\Psi : \mathfrak{B}(e) \mapsto \mathcal{H}$$

and we write $\Omega := \Psi[\mathbf{1}]$. A unitary strongly continuous representation of the Poincaré group U on $\mathcal{H}$ can be constructed, which works essentially analogous to the procedure which has been presented in [13] (compare also [24, 22]). The vector Ω is invariant under the action of U. Moreover, the spectrum of the the generator of the translations $x \mapsto U(x)$ is contained in the closed forward light cone $\bar{V}_+$.

Step 2: The construction of a Poincaré covariant Haag-Kastler net of bounded operators on $\mathcal{H}$ can be performed analogously as it has been carried out in [22]. We identify bounded operators on $\mathcal{H}$ by making use of the following proposition:

Proposition 2.1 : *For each $s \in \mathbb{R}_+$ and for each $b \in \mathfrak{B}(e,s)$, there exists a bounded operator $\pi_s(b) \in \mathfrak{B}(\mathcal{H})$ with*

$$\|\pi_s(b)\| \quad \leq \quad \|b\|$$

which is uniquely determined by the relation

$$\pi_s(b)\Psi[b_1] \quad = \quad \Psi[b\beta_{se}(b_1)]$$

for each $b_1 \in \mathfrak{B}(e)$.

Proof. The result follows by an application of the proof of [16, Theorem 10.5.5]. □

If we assume that β is strongly continuous, we expect, however, that all operators in $\underline{\mathfrak{B}}$, which are localized in the time-slice $e^{\perp}$, are multiples of the identity. But time-slice operators may be found in an appropriate *extension* of the euclidean field $(\mathfrak{B}, \beta, \eta)$.

Extension of euclidean nets. We call an euclidean field $(\hat{\mathfrak{B}}, \hat{\beta}, \hat{\eta})$ an *extension* of $(\mathfrak{B}, \beta, \eta)$ if $\hat{\mathfrak{B}}(\mathscr{U}) \supset \mathfrak{B}(\mathscr{U})$ holds true for each bounded and convex set $\mathscr{U} \subset \mathbb{R}^d$, and $\hat{\beta}_h|_{\mathfrak{B}} = \beta_h$, $\hat{\eta}|_{\mathfrak{B}} = \eta$, for each euclidean transformation h.

Indeed, there is a natural extension $(\hat{\mathfrak{B}}, \hat{\beta}, \hat{\eta})$ of the euclidean field $(\mathfrak{B}, \beta, \eta)$: Let $(\mathcal{K}, \tau, E)$ be the GNS triple of η. We introduce a topology on $\mathfrak{B}$ by semi norms

$$||b||_{\psi} \quad := \quad ||\tau(b)\psi||$$

with $\psi \in \mathcal{K}$. We denote by $\hat{\mathfrak{B}}$ the closure of $\mathfrak{B}$ within this topology and $\hat{\mathfrak{B}}$ is a W*-algebra, isomorphic to the von Neumann algebra $\tau(\mathfrak{B})''$.[2] Obviously, the homomorphism β can be extended to a homomorphism $\hat{\beta}$ form $\mathrm{E}(d)$ into the automorphism group of $\hat{\mathfrak{B}}$ and η can be extended to a reflexion positive euclidean invariant regular state $\hat{\eta}$ on $\hat{\mathfrak{B}}$.

For a subset $\mathscr{V}$ of the hyperplane $e^{\perp}$ we introduce the algebra $\hat{\mathfrak{B}}(\mathscr{V})$ of *time zero operators* which is given by the intersection

$$\hat{\mathfrak{B}}(\mathscr{V}) \quad := \quad \bigcap_{s \in \mathbb{R}_+} \hat{\mathfrak{B}}([0,s)e \times \mathscr{V}) \quad .$$

A Hilbert space $\hat{\mathcal{H}}$ can be constructed from the extended euclidean field $(\hat{\mathfrak{B}}, \hat{\beta}, \hat{\eta})$ by Step 1 and it is isomorphic to $\mathcal{H}$. Hence we may identify both spaces in the subsequent, i.e. $\hat{\mathcal{H}} = \mathcal{H}$. Analogously to the analysis, carried out in [22], we get:

Proposition 2.2 : *There exists a *-representation π of the time-zero algebra $\hat{\mathfrak{B}}(e^{\perp})$ on $\mathcal{H}$. Which is uniquely determined by the relation*

$$\pi(b)\Psi[b_1] \quad = \quad \Psi[bb_1]$$

for each $b_1 \in \hat{\mathfrak{B}}(e)$.

[2] For a algebra $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$, we write $\mathfrak{M}'$ for the commutatnt of $\mathfrak{M}$, i.e. the algebra of operators in $\mathfrak{B}(\mathcal{H})$ commuting with those in $\mathfrak{M}$.

For a double cone $\mathscr{O}$, we define $\mathfrak{A}(\mathscr{O})$ to be the C*-algebra of operators on $\mathcal{H}$ which is generated by all operators

$$\Pi[f,b] \quad := \quad \int \mathrm{d}g \; f(g) \; U(g)\pi(b)U(g)^* \tag{2}$$

with $b \in \hat{\mathfrak{B}}(\mathscr{V})$, $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$, such that $g\mathscr{V} \subset \mathscr{O}$ for each Poincaré transformation g in the support of f. The prescription

$$\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$$

is an isotonous net of C*-algebras and by putting $\alpha_g := \mathrm{Ad}(U(g))$, for each $g \in \mathrm{P}_+^\uparrow$, and $\omega := \langle \Omega, (\cdot)\Omega \rangle$ we obtain quantum field according to [22]:

Theorem 2.3 : *Let $(\mathfrak{B}, \beta, \eta)$ be a euclidean field. Then the triple $(\mathfrak{A}, \alpha, \omega)$, constructed above, is a $\mathrm{P}_+^\uparrow$-covariant quantum field.*

3 On the Short-distance analysis of field theories

We briefly review the concept of scaling algebras which has been invented by D. Buchholz and R. Verch [4, 5, 6, 7, 8, 9].

Taking scaling limits. In order to label the scaling limits, in a elegant manner, we introduce here the notion *limit functional.*

Let $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ be C*-algebra of all bounded functions on $\mathbb{R}_+$ and the closed two-sided ideal $\mathscr{F}_0(\mathbb{R}_+)$ in $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ which is generated by functions $f \in \mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ with $\lim_{\lambda \to 0} f(\lambda) = 0$. Then we build the quotient C*-algebra

$$\mathrm{C}(\mathbb{R}_+) \quad = \quad \mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)/\mathscr{F}_0(\mathbb{R}_+) \quad ,$$

the *corona algebra.* Writing $\mathrm{Sp}[\mathfrak{C}]$ for the spectrum of an abelian C*-algebra $\mathfrak{C}$, the corona algebra $\mathrm{C}(\mathbb{R}_+)$ can by interpreted as the algebra of functions which are supported on $\mathrm{Sp}[\mathrm{C}(\mathbb{R}_+)] = \mathrm{Sp}[\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)] \backslash \mathrm{Sp}[\mathscr{F}_0(\mathbb{R}_+)]$. We claim here that $\mathrm{Sp}[\mathscr{F}_0(\mathbb{R}_+)]$ is not homoeomorphic to $\mathbb{R}_+$ since $\mathscr{F}_0(\mathbb{R}_+)$ contains also functions which are discontinuous on $\mathbb{R}_+$. The set of states $\mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$ on $\mathrm{C}(\mathbb{R}_+)$ are called *limit functionals* and can be identified with the set of states $\mathfrak{S}[\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)]$ on $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ which annihilate the ideal $\mathscr{F}_0(\mathbb{R}_+)$. The reason why the states on $\mathrm{C}(\mathbb{R}_+)$ are called limit functionals becomes clear by looking at a function $f \in \mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ for which $f_0 = \lim_{\lambda \to 0} f(\lambda)$ exists. Namely, for each functional $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$ the expectation value $\langle \zeta, f \rangle = f_0$ coincides with the limit of f for $\lambda \to 0$ since ζ vanishes on $f - f_0\mathbf{1} \in \mathscr{F}_0(\mathbb{R}_+)$. A limit functional, ore more general, a state ξ on $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ can be regarded as a measure on the compact Hausdorff space $\mathrm{Sp}[\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)]$ and we write sometimes

$$\langle \xi, f\rangle \quad = \quad \int \mathrm{d}\xi(\lambda)\ f(\lambda)$$

in a suggestive manner.

Scaling limits for the euclidean fields. The *scaling algebras* are given as follows: Let $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+, \mathfrak{B})$ be the C*-algebra of bounded $\mathfrak{B}$-valued functions on $\mathbb{R}_+$ then the prescription which is given according to

$$(\underline{\beta}_h \mathbf{b})(\lambda) \quad := \quad \beta_{\lambda\circ h\circ\lambda^{-1}}\mathbf{b}(\lambda)$$

for each euclidean transformation h, yields an action $\underline{\beta}$ of the euclidean group by automorphisms on $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+, \mathfrak{B})$. In order to select the admissible *orbits* of renormalization group transformations in $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+, \mathfrak{B})$, we consider the C*-subalgebra $\underline{\mathfrak{B}}$ in $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+, \mathfrak{B})$ on which $\underline{\beta}$ is strongly continuous. In particular, for each bounded convex set $\mathscr{U}$ we denote by $\underline{\mathfrak{B}}(\mathscr{U})$ the C*-subalgebra in $\underline{\mathfrak{B}}$ which is generated by elements $\mathbf{b} \in \underline{\mathfrak{B}}$ with $\mathbf{b}(\lambda) \in \mathfrak{B}(\lambda\mathscr{U})$. This definition implies that $\underline{\beta}$ is covariant, i.e. $\underline{\beta}_h$ maps $\underline{\mathfrak{B}}(\mathscr{U})$ onto $\underline{\mathfrak{B}}(h\mathscr{U})$. In other words, the pair $(\underline{\mathfrak{B}}, \underline{\beta})$ is a euclidean net of C*-algebras.

We are now prepared to build for a given euclidean field $(\mathfrak{B}, \beta, \eta)$ the corresponding scaling limits be means of limit functionals: For a given limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$ the *scaling limit state* $\underline{\eta}_\zeta$ on the scaling algebra $\underline{\mathfrak{B}}$ is given according to the prescription

$$\langle \underline{\eta}_\zeta, \mathbf{b}\rangle \quad := \quad \int \mathrm{d}\zeta(\lambda)\ \langle \eta, \mathbf{b}(\lambda)\rangle$$

for $\mathbf{b} \in \underline{\mathfrak{B}}$. Of course, we can build the state $\underline{\eta}_\xi$ for any state ξ on $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$:

Proposition 3.1 : *Let $(\mathfrak{B}, \beta, \eta)$ be a euclidean field. Then for each state ξ on $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$ the triple $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\xi)$ is a euclidean field, i.e. $\underline{\eta}_\xi$ is a reflexion positive and euclidean invariant regular state on the scaling algebra.*

Proof. Since η is a euclidean invariant state, we conclude for each $h \in \mathrm{E}(d)$ and for each $\lambda \in \mathbb{R}_+$

$$\langle \eta, \beta_{\lambda\circ h\circ\lambda^{-1}}\mathbf{b}(\lambda)\rangle \quad = \quad \langle \eta, \mathbf{b}(\lambda)\rangle$$

for each element $\mathbf{b} \in \underline{\mathfrak{B}}$ of the euclidean scaling algebra. Applying the functional ξ to both sides yields

$$\begin{aligned}\langle \underline{\eta}_\xi, \underline{\beta}_g \mathbf{b}\rangle &= \int \mathrm{d}\xi(\lambda)\ \langle \eta, \beta_{\lambda\circ g\circ\lambda^{-1}}\mathbf{b}(\lambda)\rangle \\ &= \int \mathrm{d}\xi(\lambda)\ \langle \eta, \mathbf{b}(\lambda)\rangle \\ &= \langle \underline{\eta}_\xi, \mathbf{b}\rangle\end{aligned}$$

and hence $\underline{\eta}_\xi$ is euclidean invariant. Let $e \in S^{d-1}$ be a euclidean time direction and let $\mathbf{b} \in \underline{\mathfrak{B}}(e)$ be localized in the half space $\mathbb{R}_+ e + e^\perp$. Then we obtain from the reflexion positivity of η that

$$\langle \eta, j_e(\mathbf{b}(\lambda))\mathbf{b}(\lambda)\rangle \ \geq \ 0$$

since $\mathbf{b}(\lambda) \in \mathfrak{B}(e)$ is localized in the half space $\mathbb{R}_+ e + e^\perp$ for each λ. Since ξ is a positive functional, we get

$$\langle \underline{\eta}_\zeta, j_e(\mathbf{b})\mathbf{b}\rangle \ = \ \int \mathrm{d}\xi(\lambda)\ \langle \eta, j_e(\mathbf{b}(\lambda))\mathbf{b}(\lambda)\rangle \ \geq \ 0$$

and the reflexion positivity for $\underline{\eta}_\xi$ follows.

Finally, regularity holds true for η since $\underline{\beta}$ is strongly continuous. Namely, for $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \underline{\mathfrak{B}}$ we have

$$\begin{aligned}\lim_{h\to 1} |\langle \underline{\eta}_\xi, \mathbf{b}_1[\underline{\beta}_h(\mathbf{b}_2) - \mathbf{b}_2]\mathbf{b}_3\rangle| &\leq \ \|\mathbf{b}_1\|\ \|\mathbf{b}_3\|\ \lim_{h\to 1}\|\underline{\beta}_h(\mathbf{b}_2) - \mathbf{b}_2\| \\ &= \ 0\end{aligned}$$

and the regularity follows. □

Remark. Taking scaling limits by making use of limit functionals is slightly more general than the method of taking subnets as it has been used by D. Buchholz and R. Verch. We make some more detailed comments on this fact in Appendix A.

Quantum fields, constructed from the euclidean scaling limits. According to Proposition 3.1 the triple $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$ is an euclidean field. We recall here the procedure of Step 1 and Step 2, given in the previous section, in order to fix our notations.

Step 1': A Hilbert space $\mathcal{H}_\zeta$ and a linear map

$$\Psi_\zeta : \underline{\mathfrak{B}}(e) \to \mathcal{H}_\zeta$$

can be constructed from $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$, where Ψ_ζ is uniquely determined by

$$\langle \Psi_\zeta[\mathbf{b}_1], \Psi_\zeta[\mathbf{b}_2] \rangle \quad = \quad \langle \underline{\eta}_\zeta, j_e(\mathbf{b}_1)\mathbf{b}_2 \rangle$$

for each $\mathbf{b}_1, \mathbf{b}_2 \in \underline{\mathfrak{B}}(e)$. We obtain a unitary strongly continuous representation U_ζ of the Poincaré group on $\mathcal{H}_\zeta$ by [13], with invariant vector $\Omega_\zeta = \Psi_\zeta[\mathbf{1}]$.

Step 2': As already mentioned, the strong continuity of the group homomorphism $\underline{\beta}$ might cause the problem that all operators in $\underline{\mathfrak{B}}$ which are localized in the time-slice $e^\perp$ are multiples of the identity. Thus we wish to find an extension $(\hat{\underline{\mathfrak{B}}}, \hat{\underline{\beta}}, \hat{\underline{\eta}}_\zeta)$ of the euclidean net $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$ such that non trivial time slice operators may be found in there.

As we are going to show in Appendix D (Lemma B.1), there indeed exists an extension $(\hat{\underline{\mathfrak{B}}}, \hat{\underline{\beta}}, \hat{\underline{\eta}}_\zeta)$ of the euclidean field $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$ with the following property: Let $\mathfrak{J}_\zeta$ be the ideal which is annihilated by the GNS representation of $\hat{\underline{\eta}}_\zeta$. Then the algebra $\hat{\underline{\mathfrak{B}}}/\mathfrak{J}_\zeta$ is the extension of $\underline{\mathfrak{B}}/\mathfrak{J}_\zeta$ which can be obatained by the completion procedure in Step 2 of the previeous section. In particular, the algebra $\hat{\underline{\mathfrak{B}}}$ is independent of the limit functional ζ.

Due to Proposition 2.1, for an operators $\mathbf{b} \in \underline{\mathfrak{B}}(e,s)$, localized near the time-slice $e^\perp$, a bounded operator $\pi_{(s,\zeta)}(\mathbf{b})$ on $\mathcal{H}_\zeta$ is given by

$$\pi_{(s,\zeta)}(\mathbf{b})\Psi_\zeta[\mathbf{b}_1] \quad := \quad \Psi_\zeta[\mathbf{b}\underline{\beta}_{se}\mathbf{b}_1]$$

and by Proposition 2.2, for a time slice operator $\mathbf{b}_0 \in \hat{\underline{\mathfrak{B}}}(e^\perp)$, a bounded operator $\pi_\zeta(\mathbf{b}_0)$ on $\mathcal{H}_\zeta$ is defined by

$$\pi_\zeta(\mathbf{b}_0)\Psi_\zeta[\mathbf{b}_1] \quad := \quad \Psi_\zeta[\mathbf{b}_0\mathbf{b}_1] \ .$$

Thus we can build a Haag-Kastler net: For a double cone $\mathscr{O}$, we define $\mathfrak{A}_\zeta(\mathscr{O})$ to be the C*-subalgebra in $\mathfrak{B}(\mathcal{H}_\zeta)$ which is generated by all operators

$$\Pi_\zeta[f, \mathbf{b}] \quad = \quad \int \mathrm{d}g \ f(g) \ U_\zeta(g)\pi_\zeta(\mathbf{b})U_\zeta(g)^*$$

with $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}(\mathscr{V})$, $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$, such that $g\mathscr{V} \subset \mathscr{O}$ for each Poincaré transformation g in the support of f. The prescription

$$\mathscr{O} \mapsto \mathfrak{A}_\zeta(\mathscr{O})$$

is an isotonous net of C*-algebras and by putting $\alpha_{(\zeta,g)} := \mathrm{Ad}(U_\zeta(g))$, for each $g \in \mathrm{P}_+^\uparrow$, and $\omega_\zeta := \langle \Omega_\zeta, (\cdot)\Omega_\zeta \rangle$ we obtain the quantum field $(\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$.

Scaling limits for the constructed Haag-Kastler net. The scaling algebra $\underline{\mathfrak{A}}$ can be expressed in terms of the time-zero algebras $\hat{\underline{\mathfrak{B}}}(\mathscr{V})$ of the extended net, introduced in previous paragraph. Each operator $\mathbf{b}$ can be identified with a map $\mathbf{b} : \mathbb{R}_+ \to \hat{\mathfrak{B}}(e^\perp)$ (Lemma B.2) which assigns to each scaling parameter a time slice operator in $\hat{\mathfrak{B}}(e^\perp)$. The scaling algebra $\underline{\mathfrak{A}}$ is generated by all functions

$$\underline{\Pi}[f, \mathbf{b}] : \lambda \quad \mapsto \quad \underline{\Pi}[f, \mathbf{b}](\lambda) \; := \int \mathrm{d}g \; f(g) \; U_\lambda(g) \pi(\mathbf{b}(\lambda)) U_\lambda(g)^*$$

where $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$ is a smooth function on the Poincaré group with compact support and $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}(e^\perp)$ is a time slice operator. Here, U_λ is the scaled representation of the Poincaré group on $\mathcal{H}$, which is defined according to $U_\lambda(g) := U(\lambda \circ g \circ \lambda^{-1})$ for each Poincaré transformation g.

As also described in the introduction, a group homomorphism $\underline{\alpha}$ from the Poincaré group into the automorphism group of the scaling algebra $\underline{\mathfrak{A}}$, acting covariantly on the net $\mathscr{O} \mapsto \underline{\mathfrak{A}}(\mathscr{O})$, is simply defined according to

$$(\underline{\alpha}_g \mathbf{a})(\lambda) \quad := \quad \alpha_{\lambda \circ g \circ \lambda^{-1}} \mathbf{a}(\lambda)$$

for each $\mathbf{a} \in \underline{\mathfrak{A}}$.

Of course, we have to ensure that the algebra $\underline{\mathfrak{A}}$ contains renormalization group orbits with the correct scaling property in configuration space as well as in momentum space:

Proposition 3.2 : *The group homomorphism $\underline{\alpha}$ is strongly continuous on $\underline{\mathfrak{A}}$:*

$$\lim_{g \to 1} \| \mathbf{a} - \underline{\alpha}_g \mathbf{a} \| \quad = \quad 0$$

for each $\mathbf{a} \in \underline{\mathfrak{A}}$.

Proof. We first check the continuety for all generators $\underline{\Pi}[f, \mathbf{b}]$, where $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$ is a smooth function on the Poincaré group with compact support and $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}(e^\perp)$ is a time slice operator:

$$\begin{aligned} & \lim_{g\to 1} \|\underline{\Pi}[f,\mathbf{b}] - \underline{\alpha}_g \underline{\Pi}[f,\mathbf{b}]\| \\ = & \lim_{g\to 1} \|\underline{\Pi}[f,\mathbf{b}] - \underline{\Pi}[f\circ g^{-1},\mathbf{b}]\| \\ = & \lim_{g\to 1} \|\underline{\Pi}[f - f\circ g^{-1},\mathbf{b}]\| \\ \leq & \sup_{\lambda\in\mathbb{R}_+} \|\mathbf{b}(\lambda)\| \lim_{g\to 1} \int \mathrm{d}g' \ |f(g') - f(g^{-1}g')| \\ = & \ 0 \ . \end{aligned}$$

Now let $\mathbf{a}_1, \mathbf{a}_2$ be two operators with $\lim_{g\to 1} \|\mathbf{a}_j - \underline{\alpha}_g \mathbf{a}_j\| = 0$ for $j = 1, 2$ then we obtain

$$\begin{aligned} & \lim_{g\to 1} \|\mathbf{a}_1\mathbf{a}_2 - \underline{\alpha}_g(\mathbf{a}_1\mathbf{a}_2)\| \\ \leq & \ \|\mathbf{a}_1\| \lim_{g\to 1} \|\mathbf{a}_2 - \underline{\alpha}_g(\mathbf{a}_2)\| + \|\mathbf{a}_2\| \lim_{g\to 1} \|\mathbf{a}_1 - \underline{\alpha}_g(\mathbf{a}_1)\| \\ = & \ 0 \end{aligned}$$

which implies that strong continuety is valid for all finite linear combinations of products of generators. For each operator $\mathbf{a} \in \underline{\mathfrak{A}}$ and for each $\epsilon > 0$ we can find a finite linear combination of products of generators $\mathbf{a}_\epsilon$ such that $\|\mathbf{a} - \mathbf{a}_\epsilon\| < \epsilon/2$. Thus we get

$$\begin{aligned} \lim_{g\to 1} \|\mathbf{a} - \underline{\alpha}_g(\mathbf{a})\| & \leq 2\|\mathbf{a} - \mathbf{a}_\epsilon\| + \lim_{g\to 1} \|\mathbf{a}_\epsilon - \underline{\alpha}_g(\mathbf{a}_\epsilon)\| \\ & < \epsilon \end{aligned}$$

and hence $\lim_{g\to 1} \|\mathbf{a} - \underline{\alpha}_g(\mathbf{a})\| = 0$ for all $\mathbf{a} \in \underline{\mathfrak{A}}$. $\square$

For a given limit functional ζ we build the scaling limit $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$ of the quantum field $(\mathfrak{A}, \alpha, \omega)$ in the following manner: A vacuum state $\underline{\omega}_\zeta$ is given according to the prescription

$$\langle \underline{\omega}_\zeta, \mathbf{a}\rangle := \int \mathrm{d}\zeta(\lambda) \ \langle \omega, \mathbf{a}(\lambda)\rangle$$

for each $\mathbf{a} \in \underline{\mathfrak{A}}$. Let $\mathfrak{J}_\zeta$ be the two sided ideal in $\underline{\mathfrak{A}}$ which is annihilated by the GNS representation of $\underline{\omega}_\zeta$. We build the quotient C*-algebra

$$\underline{\mathfrak{A}}_\zeta := \underline{\mathfrak{A}}/\mathfrak{J}_\zeta$$

and we denote by $\mathbf{q}_\zeta$ the corresponding canonical projection onto the quotient. The group homomorphism $\underline{\alpha}$ can be lifted to a group homomorphism $\underline{\alpha}_\zeta$ from the Poincaré group into the automorphism group of $\underline{\mathfrak{A}}_\zeta$ by

$$\underline{\alpha}_{(\zeta,g)} \circ \mathbf{q}_\zeta \quad = \quad \mathbf{q}_\zeta \circ \underline{\alpha}_g$$

for each Poincaré trnsformation g. As a consequence the assignment

$$\mathscr{O} \quad \mapsto \quad \underline{\mathfrak{A}}_\zeta(\mathscr{O}) \quad := \quad \mathbf{q}_\zeta(\underline{\mathfrak{A}}(\mathscr{O}))$$

together with the group homomorphism $\underline{\alpha}_\zeta$ is a Poincaré covariant Haag-Kastler net.

The main result. We are now prepared to show that the subsequent two step procedures applied to a given euclidean field $(\mathfrak{B}, \beta, \eta)$, lead to equivalent results.

(1) For a given limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$ we first build the euclidean field $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$ and then the corresponding quantum field theory model $(\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$.

(2) On the other hand, we first build the quantum field theory model $(\mathfrak{A}, \alpha, \omega)$ and then we construct the scaling limit $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$ with respect to a limit point $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$.

Theorem 3.3 : *For each limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$, the quantum fields $(\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$ and $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$ are equivalent.*

The complete proof of Theorem 3.3 is given in the Appendix D and we only describe the main idea of it here.

Sketch of the proof: In the first step, a Poincaré covariant representation $\underline{\pi}_\zeta$ of the scaling algebra $\underline{\mathfrak{A}}$ on a Hilbert space $\underline{\mathcal{H}}_\zeta$ is constructed and it can be shown that $\underline{\pi}_\zeta$ is equivalent to the GNS representation of the scaling limit state $\underline{\omega}_\zeta$. In the second step, an isometry $\mathbf{u}_\zeta$ from the Hilbert space $\mathcal{H}_\zeta$ to $\underline{\mathcal{H}}_\zeta$ is constructed such that

$$\mathbf{u}_\zeta \Pi_\zeta[f, \mathbf{b}] \mathbf{u}_\zeta^* \quad = \quad \underline{\pi}_\zeta(\underline{\Pi}[f, \mathbf{b}]) \tag{3}$$

holds true for each smooth function with compact support f on the Poincaré group and for each time zero operator $\mathbf{b}$ in the extended scaling algebra $\hat{\underline{\mathfrak{B}}}(e^\perp)$. Note, that the operator $\Pi_\zeta[f, \mathbf{b}]$ is contained in the C*-algebra $\mathfrak{A}_\zeta$, corresponding to procedure *(1)*, whereas $\underline{\pi}_\zeta(\underline{\Pi}[f, \mathbf{b}])$ can be identified with an operator in the C*-algebra $\underline{\mathfrak{A}}_\zeta$ corresponding to procedure *(2)*. As a consequence the map

$$\iota_\zeta : \mathbf{a} \quad \mapsto \quad \mathbf{u}_\zeta \mathbf{a} \mathbf{u}_\zeta^*$$

yields an isomorphism of the algebras $\mathfrak{A}_\zeta$ and $\underline{\mathfrak{A}}_\zeta$ which respects the net structure, $\iota_\zeta(\mathfrak{A}_\zeta(\mathscr{O})) = \underline{\mathfrak{A}}_\zeta(\mathscr{O})$ for each double cone $\mathscr{O}$. Moreover, ι_ζ intertwines the automorphisms α_ζ and $\underline{\alpha}_\zeta$, i.e. $\iota_\zeta \circ \alpha_\zeta = \underline{\alpha}_\zeta \circ \iota_\zeta$, and maps the vacuum state $\underline{\omega}_\zeta$ to ω_ζ, i.e. $\underline{\omega}_\zeta \circ \iota_\zeta = \omega_\zeta$.

The crucial point in order to prove Equation (3) is to establish the fact that $\mathbf{u}_\zeta$ intertwines the corresponding representations of the Poincaré group. More precisely, the Hilbert space $\mathcal{H}_\zeta$ (respectively $\underline{\mathcal{H}}_\zeta$), carry a strongly continuous unitary representation U_ζ (respectively $\underline{U}_\zeta$) of the Poincaré group and one has to show that

$$\mathbf{u}_\zeta U_\zeta(g)\mathbf{u}_\zeta^* \quad = \quad \underline{U}_\zeta(g)$$

holds true for each Poincaré transformation g. This intertwiner property can be verified as follows: Let B be the generator of a Poincaré transformation. Then there is a dense subspace $\mathcal{D} \subset \underline{\mathcal{H}}_\zeta$ such that for each pair ψ_1, ψ_2 the functions

$$\underline{F} : z \mapsto \langle \psi_1, \underline{U}_\zeta(\exp(zB))\psi_2 \rangle$$

$$F : z \mapsto \langle \mathbf{u}_\zeta^*\psi_1, U_\zeta(\exp(zB))\mathbf{u}_\zeta^*\psi_2 \rangle$$

are holomorphic within a strip $\mathbb{R} + \mathrm{i}I$, where I is some connected open subset in $\mathbb{R}$. Within the pure imaginary points, both functions can be expressed explicitly in terms of euclidean correlation functions within the scaling limit state $\underline{\eta}_\zeta$ and one finds that

$$\underline{F}(\mathrm{i}\tau) \quad = \quad F(\mathrm{i}\tau)$$

is valid for each $\tau \in I$. Hence one concludes $F = \underline{F}$ and the interwiner property follows since $\mathcal{D}$ is dense in $\underline{\mathcal{H}}_\zeta$.

4 Concluding remarks

We have proven that the quantum field theories, which can be reconstructed from euclidean scaling limit theories, are scaling limit theories of the quantum field theory which can be reconstructed from the underlying euclidean field theory.

This fact leads us into a comfortable position which can be also motivated by the consideration of euclidean field theories with cutoffs. Usually, to a regularized euclidean model the construction scheme [22] cannot even by applied. On the other hand, the euclidean counterpart of the analysis by D. Buchholz and R. Verch [8, 5] can be formulated for euclidean field theory models in the C*-setting as we have discussed during this paper. The procedure, given in Section 3, is quite general and it still makes sense also for regularized euclidean models with a infra-red and an ultra-violet cutoff.

We expect that scaling limit theories of euclidean field theories within a finite volume are essentially independent of the volume cutoff. We propose to regard a finite volume euclidean field theory in d dimensions as a field theory on the scaled d-sphere $rS^d \subset \mathbb{R}^{d+1}$, where r is the volume cutoff. The corresponding euclidean net $\mathfrak{B}$ carries a covariant action of the $d+1$-dimensional rotation group $\mathrm{O}(d+1)$ and the functional η is invariant under this action. For a point $x_0 \in rS^d$, the stabilizer subgroup of x_0 in $\mathrm{O}(d+1)$ is isomorphic to $\mathrm{O}(d)$. If the scaling limit procedure is performed at x_0, the invariance under the stabilizer subgroup should remain as a $\mathrm{O}(d)$ invariance within the scaling limit. The translation invariance should then enter from the fact that η is invariant under the full group $\mathrm{O}(d+1)$. Note, that r is replaced by $r_\lambda = \lambda^{-1}r$ for the scaled theory. Hence one expects that the scaling limit theories are models within an infinite volume.

Keeping in mind that the minkowskian analogue of the euclidean d-sphere $rS^d \subset \mathbb{R}^{d+1}$ is the de Sitter space, it should be possible, by exploring the analytic structure of de Sitter space, to construct from a given euclidean field theory $(\mathfrak{B}, \alpha, \eta)$ on the sphere rS^d a quantum field theory $(\mathfrak{A}, \alpha, \omega)$ in de Sitter space. According to the considerations of H. J. Borchers and D. Buchholz [2] we conjecture that the reconstructed state ω fulfills the so called *geodesic KMS condition*, i.e. for any geodesic observer the state ω looks like an equilibrium state.

Acknowledgment:

I am grateful to Prof. Jakob Yngvason for supporting this investigation with hints and many ideas. This investigation is financially supported by the *Jubiläumsfond der Oesterreichischen Nationalbank* which is also gratefully acknowledged. Finally I would like to thank the Erwin Schrödinger International Institute for Mathematical Physics, Vienna (ESI) for its hospitality.

A Remarks on limit functionals

As a tool for taking scaling limits we have used the limit functional on the corona algbera $\mathrm{C}(\mathbb{R}_+) = \mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)/\mathscr{F}_0(\mathbb{R}_+)$. More general, we replace $\mathbb{R}_+$ by any partially ordered directed set Λ, and we consider the C*-algebra $\mathscr{F}_{\mathrm{b}}(\Lambda)$ of all bounded functions on Λ and the closed two-sided ideal $\mathscr{F}_0(\Lambda)$ in $\mathscr{F}_{\mathrm{b}}(\Lambda)$ which is generated by functions $f \in \mathscr{F}_{\mathrm{b}}(\Lambda)$ with $\lim_{\lambda\in\Lambda} f(\lambda) = 0$. The states $\zeta \in \mathfrak{S}[\mathrm{C}(\Lambda)]$ on the quotient C*-algebra

$$\mathrm{C}(\Lambda) \quad = \quad \mathscr{F}_{\mathrm{b}}(\Lambda)/\mathscr{F}_0(\Lambda)$$

are called the *limit functionals* with respect to Λ.

Let V be a locally convex linear space whose topology is induced by a family of semi norms $\mathbf{p} \in \mathscr{P}$. A function $\mathbf{w} : \Lambda \to V'$ from the partially ordered set Λ to the dual space V' is called *bounded* if there exists a semi norm $\mathbf{p} \in \mathscr{P}$ and a constant $K > 0$ such that

$$\sup_{\lambda\in\Lambda} |\langle \mathbf{w}(\lambda), v\rangle| \quad \leq \quad K\ \mathbf{p}(v)$$

holds true for each $v \in V$. For each limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\Lambda)]$ we obtain a linear functional $\mathbf{w}_\zeta$ on V by the prescription

$$\langle \mathbf{w}_\zeta, v\rangle \quad := \quad \int \mathrm{d}\zeta(\lambda)\ \langle \mathbf{w}(\lambda), v\rangle \quad .$$

The functional $\mathbf{w}_\zeta$ is well defined since for each $v \in V$ the function $[\lambda \mapsto \langle \mathbf{w}(\lambda), v\rangle]$ is contained in $\mathscr{F}_{\mathrm{b}}(\Lambda)$. In particular, $\mathbf{w}_\zeta$ fulfills for each $v \in V$ the estimate

$$|\langle \mathbf{w}_\zeta, v\rangle| \quad \leq \quad K\ \mathbf{p}(v)$$

and $\mathbf{w}_\zeta$ is a continuous functional on V.

As a special case, let V be a Banach space with norm $\|\cdot\|$ and let $\|\cdot\|'$ be the dual norm on V'. For each bounded function $\mathbf{w} : \Lambda \to V'$, we create new elements in V' in two different ways:

(1) According to the Banach-Alaoglu theorem, the set $\{\mathbf{w}(\lambda), \lambda \in \Lambda\}$ is precompact and there exists a partially ordered directed set J and a subnet $\iota : J \to \Lambda$ such that

$$\langle \mathbf{w}_\iota, v\rangle \quad := \quad \lim_{j\in J}\langle \mathbf{w}(\iota(j)), v\rangle$$

for each $v \in V$.

(2) We also can choose a limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\Lambda)]$ and build the linear functional $\mathbf{w}_\zeta \in V'$ as described above.

Choosing the state $\mathbf{w}_\iota$, via a subnet $\iota : J \to \Lambda$, is essentially the same as choosing a limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\Lambda)]$. More precisely:

Proposition A.1 : *For each subnet $\iota : J \to \Lambda$ for which the weak*-limit $\mathbf{w}_\iota = w^* - \lim_{j\in J} \mathbf{w}(\iota(j))$ exists in V', there exists a limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\Lambda)]$ such that*

$$\mathbf{w}_\iota \quad = \quad \mathbf{w}_\zeta \quad .$$

Proof. The subnet $\iota : J \to \Lambda$ induces a *-homomorphism $\iota^* : \mathscr{F}_{\mathrm{b}}(\Lambda) \to \mathscr{F}_{\mathrm{b}}(J)$ by $\iota^* f = f \circ \iota$. One observes that for each state $\hat{\zeta} \in \mathfrak{S}[\mathrm{C}(J)]$ on the corona algebra of J, the state $\zeta = \hat{\zeta} \circ \iota^*$ yields a state on $\mathrm{C}(\Lambda)$. Namely, let $f \in \mathscr{F}_0(\Lambda)$ then $\lim_{\lambda\in\Lambda} f(\lambda) = 0$ and thus $\lim_{j\in J} f(\iota(j)) = 0$ for each subnet $\iota : J \to \Lambda$. Hence $\iota^*\mathscr{F}_0(\Lambda) \subset \mathscr{F}_0(J)$ and the state $\zeta = \hat{\zeta} \circ \iota^*$ annihilates $\mathscr{F}_0(\Lambda)$ which implies $\zeta \in \mathfrak{S}[\mathrm{C}(\Lambda)]$. Since for each $v \in V$ the function $[j \mapsto \langle \mathbf{w}(\iota(j)) - \mathbf{w}_\iota, v\rangle]$ is contained in $\mathscr{F}_0(J)$ we conclude

$$\begin{aligned} \langle \mathbf{w}_\iota, v\rangle \quad &= \quad \int \mathrm{d}\hat{\zeta}(j)\ \langle \mathbf{w}(\iota(j)), v\rangle \\ &= \quad \int \mathrm{d}(\hat{\zeta} \circ \iota^*)(\lambda)\ \langle \mathbf{w}(\lambda), v\rangle \\ &= \quad \langle \mathbf{w}_\zeta, v\rangle \end{aligned}$$

which implies the proposition. □

B Notes on scaling limits of euclidean fields

In order to construct from the euclidean data $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$ a quantum field theory model, we discuss now an appropriate extension of the scaling algebra $\underline{\mathfrak{B}}$.

Let $(\mathcal{K}, \tau, E)$ be the GNS triple of η. According to the euclidean invariance of η, there exists a strongly continuous representation V of the euclidean group on $\mathcal{K}$ which is uniquely determined by

$$V(h)\tau(b)E \quad = \quad \tau(\beta_h b)E$$

for each $b \in \mathfrak{B}$ and for each euclidean transformation h. Let $\underline{\mathcal{K}}_o$ be the linear space, spanned by bounded functions $\underline{\psi} : \mathbb{R}_+ \to \mathcal{K}$ of the form

$$\underline{\psi}_\lambda \quad = \quad \int \mathrm{d}h\ f(h)\ V_\lambda(h)\underline{\psi}^0_\lambda$$

where $\underline{\psi}^0 : \mathbb{R}_+ \to \mathcal{K}$ is any bounded function and $V_\lambda(h) = V(\lambda \circ h \circ \lambda^{-1})$ is the scaled representation. We introduce a locally convex topology on $\underline{\mathcal{K}}_o$ which is induced by semi norms $||\cdot||_\xi$, where ξ is a state on $\mathfrak{F}_\mathrm{b}(\mathbb{R}_+)$:

$$||\underline{\psi}||^2_\xi \quad := \quad \int \mathrm{d}\xi(\lambda)\ \langle \underline{\psi}_\lambda, \underline{\psi}_\lambda \rangle \quad .$$

The closure of $\underline{\mathcal{K}}_o$ with respect to this topology is denoted by $\underline{\mathcal{K}}$. The prescription

$$[\underline{\tau}(\mathbf{b})\underline{\psi}]_\lambda \quad := \quad \tau(\mathbf{b}(\lambda))\underline{\psi}_\lambda$$

yields a faithful representation of the scaling algebra on $\underline{\mathcal{K}}$ by bounded operators $\underline{\tau}(\mathbf{b})$.

We also introduce semi norms on $\underline{\mathfrak{B}}$, namely for a state ξ on $\mathfrak{F}_\mathrm{b}(\mathbb{R}_+)$ and for a vector $\underline{\Psi}$ we introduce the semi norm $||\cdot||_{(\xi,\underline{\psi})}$ by

$$||\mathbf{b}||_{(\xi,\underline{\psi})} \quad = \quad ||\underline{\tau}(\mathbf{b})\underline{\psi}||_\xi$$

and the closure of $\underline{\mathfrak{B}}$ within this topology is denoted by $\hat{\underline{\mathfrak{B}}}$. Note that the group homomorphism $\underline{\beta}$ can be extended to a group homomorphism $\hat{\underline{\beta}}$ form the euclidean group into the automorphism group of $\hat{\underline{\mathfrak{B}}}$.

For each state ξ on $\mathscr{F}_\mathrm{b}(\mathbb{R}_+)$ there is a Hilbert space $\underline{\mathcal{K}}_\xi$ and a linear map $\mathbf{p}_\xi : \underline{\mathcal{K}} \to \underline{\mathcal{K}}_\xi$ which is uniquely determined by

$$\langle \mathbf{p}_\xi[\underline{\psi}_1], \mathbf{p}_\xi[\underline{\psi}_2] \rangle \quad = \quad \int \mathrm{d}\xi(\lambda)\ \langle \underline{\psi}_{1,\lambda}, \underline{\psi}_{2,\lambda} \rangle \quad .$$

We also use, which is sometimes convenient, the suggestive notation

$$\mathbf{p}_\xi[\underline{\psi}] \quad = \quad \int^\oplus \mathrm{d}\xi(\lambda)\ \underline{\psi}_\lambda \quad .$$

Lemma B.1 : *For each state ξ on $\mathscr{F}_\mathrm{b}(\mathbb{R}_+)$ the following statements are valid:*

(1) There exists a strongly continuous unitary representation $\underline{V}_\xi$ on $\underline{\mathcal{K}}_\xi$ of the euclidean group.

*(2) There exists a *-representation $\underline{\tau}_\xi$ on $\underline{\mathcal{K}}_\xi$ of the extended scaling algebra $\hat{\underline{\mathfrak{B}}}$ and a vector $\underline{E}_\xi \in \underline{\mathcal{K}}_\xi$ such that*

$$\langle \underline{\eta}_\xi, \mathbf{b} \rangle \quad = \quad \langle \underline{E}_\xi, \underline{\tau}_\xi(\mathbf{b})\underline{E}_\xi \rangle$$

is valid for each $\mathbf{b} \in \underline{\mathfrak{B}}$.

(3) For each $h \in \mathrm{E}(d)$ and for each $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}$ the identity

$$\underline{\tau}_\xi(\hat{\underline{\beta}}_h \mathbf{b}) \quad = \quad \underline{V}_\xi(h)\underline{\tau}_\xi(\mathbf{b})\underline{V}_\xi(h)^*$$

holds true.

Proof. We define the representation $\underline{V}_\xi$ of the euclidean group by

$$\underline{V}_\xi(h)\mathbf{p}_\xi[\underline{\psi}] \quad = \quad \int^{\oplus} \mathrm{d}\xi(\lambda)\ V_\lambda(h)\ \underline{\psi}_\lambda$$

for each $h \in \mathrm{E}(d)$ and for each $\underline{\psi} \in \underline{\mathcal{K}}$. For each $\mathbf{b} \in \underline{\mathfrak{B}}$ and for each $\underline{\psi} \in \underline{\mathcal{K}}$ the map $\mathbf{b} \mapsto \underline{\tau}(\mathbf{b})\underline{\psi}$ is continuous as a linear function form $\hat{\underline{\mathfrak{B}}}$ to $\underline{\mathcal{K}}$ since for each ξ we have

$$\|\underline{\tau}(\mathbf{b})\underline{\psi}\|_\xi \quad = \quad \|\mathbf{b}\|_{(\xi,\underline{\psi})}$$

and $\underline{\tau}$ can uniquely be extended to representation $\hat{\underline{\tau}}$ of $\hat{\underline{\mathfrak{B}}}$ on $\underline{\mathcal{K}}$. Now we define the representation $\underline{\tau}_\xi$ by

$$\underline{\tau}_\xi(\mathbf{b})\mathbf{p}_\xi[\underline{\psi}] \quad := \quad \mathbf{p}_\xi[\hat{\underline{\tau}}(\mathbf{b})\underline{\psi}]$$

for each $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}$ and for each $\underline{\psi} \in \underline{\mathcal{K}}$.

(1) We first show that $\underline{V}_\xi$ is indeed a strongly continuous representation of the euclidean group. The vectors of $\underline{\psi}_\xi$ of the form

$$\underline{\psi}_\xi \quad = \quad \int^{\oplus} \mathrm{d}\xi(\lambda) \int \mathrm{d}h\ f(h)\ V_\lambda(h)\underline{\psi}^0_\lambda \quad ,$$

where $\underline{\psi}^0 : \mathbb{R}_+ \to \mathcal{K}$ is any bounded function, span a dense subspace in $\underline{\mathcal{K}}_\xi$. We compute

$$\begin{aligned}\underline{V}_\xi(h)\underline{\psi}_\xi \quad &= \quad \int^{\oplus} \mathrm{d}\xi(\lambda) \int \mathrm{d}h'\ f(h')\ V_\lambda(hh')\underline{\psi}^0_\lambda \\ &= \quad \int^{\oplus} \mathrm{d}\xi(\lambda) \int \mathrm{d}h'\ f(h^{-1}h')\ V_\lambda(h')\underline{\psi}^0_\lambda\end{aligned}$$

which yields

$$\|\underline{V}_\xi(h)\underline{\psi}_\xi - \underline{\psi}_\xi\| \quad \leq \quad \int \mathrm{d}h'\ |f(h^{-1}h') - f(h')|\ \sup_{\lambda\in\mathbb{R}_+} \|\underline{\psi}^0_\lambda\|$$

and we conclude that

$$\lim_{h\to 1} \|\underline{V}_\xi(h)\underline{\psi}_\xi - \underline{\psi}_\xi\| \quad = \quad 0$$

holds true for all $\underline{\psi}_\xi$ in a dense subspace of $\underline{\mathcal{K}}_\xi$ and the statement *(1)* follows.

(2) The Hilbert space $\underline{\mathcal{K}}_\xi$ contains a distinguished vector $\underline{E}_\xi$

$$\underline{E}_\xi \quad = \quad \int^{\oplus} \mathrm{d}\xi(\lambda)\ E$$

which is invariant under the representation $\underline{V}_\xi$. This vector yields a state $\underline{\hat{\eta}}_\xi$ on the extended scaling algebra $\underline{\hat{\mathfrak{B}}}$ by

$$\langle \underline{\hat{\eta}}_\xi, \mathbf{b} \rangle \quad := \quad \langle \underline{E}_\xi, \underline{\tau}_\xi(\mathbf{b}) \underline{E}_\xi \rangle$$

for $\mathbf{b} \in \underline{\hat{\mathfrak{B}}}$. According to the construction of $\underline{\mathcal{K}}_\xi$ we easily observe that $\underline{\hat{\eta}}_\xi$ is an extension of $\underline{\eta}_\xi$:

$$\begin{aligned} \langle \underline{E}_\xi, \underline{\tau}_\xi(\mathbf{b}) \underline{E}_\xi \rangle &= \int \mathrm{d}\xi(\lambda) \ \langle E, \tau(\mathbf{b}(\lambda)) E \rangle \\ &= \int \mathrm{d}\xi(\lambda) \ \langle \eta, \mathbf{b}(\lambda) \rangle \\ &= \langle \underline{\eta}_\xi, \mathbf{b} \rangle \end{aligned}$$

for each $\mathbf{b} \in \underline{\mathfrak{B}}$ which implies *(2)*.

(3) Statement *(3)* can easily be verfied, from the definition of the representations $\underline{\tau}_\xi$ and $\underline{V}_\xi$.

□

The extended state $\underline{\hat{\eta}}_\xi$ is euclidean invariant, reflexion positive, and regular. In particular, the regularity follows from the fact that $\underline{V}_\xi$ is a strongly continuous representation of the euclidean group. Hence the triple $(\underline{\hat{\mathfrak{B}}}, \underline{\hat{\beta}}, \underline{\hat{\eta}}_\xi)$ is a euclidean field which extends $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\xi)$.

Lemma B.2 : *The time zero algebra $\underline{\hat{\mathfrak{B}}}(e^\perp)$ can be identified with the C^*-algebra generated by bounded functions $\mathbf{b} : \lambda \mapsto \mathbf{b}(\lambda) \in \hat{\mathfrak{B}}(\lambda \mathscr{V})$ mapping each $\lambda \in \mathbb{R}_+$ into the algebra $\hat{\mathfrak{B}}(\lambda \mathscr{V})$ for some convex and bounded region $\mathscr{V} \subset e^\perp$.*

Proof. For each $\lambda \in \mathbb{R}_+$ the representation $\underline{\tau}_\lambda$ is a representation of $\underline{\hat{\mathfrak{B}}}$ on $\mathcal{K}$ since we have for each $\mathbf{b} \in \underline{\mathfrak{B}}$

$$\underline{\tau}_\lambda(\mathbf{b}) \mathbf{p}_\lambda[\underline{\psi}] \quad = \quad \tau(\mathbf{b}(\lambda)) \underline{\psi}_\lambda \quad .$$

We have assumed that τ is a faithful representation and we may define for each time zero operator $\mathbf{b} \in \underline{\hat{\mathfrak{B}}}(\mathscr{V})$, $\mathscr{V} \subset e^\perp$, the operator

$$\mathbf{b}(\lambda) \quad := \quad \tau^{-1}(\underline{\tau}_\lambda(\mathbf{b}))$$

which is contained in $\hat{\mathfrak{B}}$. According to the localizing property of $\mathbf{b}$ we conclude

$$\mathbf{b}(\lambda) \quad \in \quad \bigcap_{s \in \mathbb{R}_+} \hat{\mathfrak{B}}([0, \lambda s) e \times \lambda \mathscr{V}) \quad = \quad \hat{\mathfrak{B}}(\lambda \mathscr{V}) \quad .$$

Since the W*-algebra $\hat{\mathfrak{B}}$ is closed in the strong operator topology, for a given function $\mathbf{b} : \lambda \mapsto \mathbf{b}(\lambda) \in \hat{\mathfrak{B}}(\lambda\mathscr{V})$ there is a net of operators $(\mathbf{b}_j)_{j\in J}$ in $\underline{\mathfrak{B}}$, converging in $\hat{\underline{\mathfrak{B}}}$, such that

$$\lim_{j\in J} \|\mathbf{b}_j - \mathbf{b}\|_{(\lambda,\underline{\psi})} \quad = \quad \lim_{j\in J} \|\tau(\mathbf{b}_j(\lambda) - \mathbf{b}(\lambda))\underline{\psi}_\lambda\| \quad = \quad 0$$

for each $\underline{\psi} \in \underline{\mathcal{K}}$, pointwise in $\lambda \in \mathbb{R}_+$. □

C Notes on scaling limits of quantum fields constructed form euclidean data

Let $\underline{\mathcal{H}}_o$ be the linear space, spanned by bounded functions $\underline{\psi} : \mathbb{R}_+ \to \mathcal{H}$ of the form

$$\underline{\psi}_\lambda \quad = \quad \int \mathrm{d}g \ f(g) \ U_\lambda(g)\underline{\psi}^0_\lambda$$

where $\underline{\psi}^0 : \mathbb{R}_+ \to \mathcal{H}$ is any bounded function and $U_\lambda(g) = U(\lambda \circ g \circ \lambda^{-1})$ is the scaled representation. We introduce a locally convex topology on $\underline{\mathcal{H}}_o$ which is induced by semi norms $\|\cdot\|_\xi$, where ξ is a state on $\mathfrak{F}_\mathrm{b}(\mathbb{R}_+)$:

$$\|\underline{\psi}\|^2_\xi \quad := \quad \int \mathrm{d}\xi(\lambda) \ \langle \underline{\psi}_\lambda, \underline{\psi}_\lambda \rangle \quad .$$

The closure of $\underline{\mathcal{H}}_o$ with respect to this topology is denoted by $\underline{\mathcal{H}}$. The prescription

$$[\underline{\pi}(\mathbf{a})\underline{\psi}]_\lambda \quad := \quad \mathbf{a}(\lambda)\underline{\psi}_\lambda$$

yields a faithful representation $\underline{\pi}$ of the scaling algebra $\underline{\mathfrak{A}}$ on $\underline{\mathcal{H}}$ by bounded operators.

As in the euclidean case, for each state ξ on $\mathscr{F}_\mathrm{b}(\mathbb{R}_+)$ there is a Hilbert space $\underline{\mathcal{H}}_\xi$ and a linear map $\mathbf{q}_\xi : \underline{\mathcal{H}} \to \underline{\mathcal{H}}_\xi$ which is uniquely determined by

$$\langle \mathbf{q}_\xi[\underline{\psi}_1], \mathbf{q}_\xi[\underline{\psi}_2] \rangle \quad = \quad \int \mathrm{d}\xi(\lambda) \ \langle \underline{\psi}_{1,\lambda}, \underline{\psi}_{2,\lambda} \rangle$$

and we also write

$$\mathbf{q}_\xi[\underline{\psi}] \quad = \quad \int^\oplus \mathrm{d}\xi(\lambda) \ \underline{\psi}_\lambda \quad .$$

Analogously to Lemma B.1 in the euclidean case, we obtain for each state ξ on $\mathscr{F}_\mathrm{b}(\mathbb{R}_+)$ a strongly continuous representation $\underline{U}_\xi$ of the

Poincaré group as well as a representation $\underline{\pi}_\xi$ of the scaling algebra $\underline{\mathfrak{A}}$ on $\underline{\mathcal{H}}_\xi$ such that

$$\underline{\pi}_\xi(\alpha_g \mathbf{a}) = \underline{U}_\xi(g)\underline{\pi}_\xi(\mathbf{a})\underline{U}_\xi(g)^*$$

where $\underline{\pi}_\xi$ and $\underline{U}_\xi$ are given by

$$\underline{\pi}_\xi(\mathbf{a})\mathbf{q}_\xi[\underline{\psi}] = \int^\oplus \mathrm{d}\xi(\lambda)\ \mathbf{a}(\lambda)\underline{\psi}_\lambda$$

$$\underline{U}_\xi(g)\mathbf{q}_\xi[\underline{\psi}] = \int^\oplus \mathrm{d}\xi(\lambda)\ U_\lambda(g)\ \underline{\psi}_\lambda \quad .$$

We are now prepared to formulate a lemma which turns out to be very useful for our subsequent analysis and which relates the representation $\underline{\pi}_\zeta$ to the GNS triple $(\mathcal{H}_{\underline{\omega}_\zeta}, \pi_{\underline{\omega}_\zeta}, \Omega_{\underline{\omega}_\zeta})$ of the scaling limit $\underline{\omega}_\zeta$:

Lemma C.1 : *For each limit functional ζ there exists an isometry*

$$\mathbf{v}_\zeta : \mathcal{H}_{\underline{\omega}_\zeta} \to \underline{\mathcal{H}}_\zeta$$

which intertwines the GNS representation $\pi_{\underline{\omega}_\zeta}$ of the scaling limit $\underline{\omega}_\zeta$ and the representation of $\underline{\pi}_\zeta$ on the scaling algebra:

$$\mathbf{v}_\zeta \pi_{\underline{\omega}_\zeta}(\mathbf{a}) = \underline{\pi}_\zeta(\mathbf{a})\mathbf{v}_\zeta$$

for each $\mathbf{a} \in \underline{\mathfrak{A}}$.

Proof. An isometry

$$\mathbf{v}_\zeta : \mathcal{H}_{\underline{\omega}_\zeta} \to \underline{\mathcal{H}}_\zeta$$

is given by the prescription

$$\mathbf{v}_\zeta[\pi_{\underline{\omega}_\zeta}(\mathbf{a})\Omega_{\underline{\omega}_\zeta}] = \underline{\pi}_\zeta(\mathbf{a})\underline{\Omega}_\zeta$$

where $\underline{\Omega}_\zeta$ is the equivalence class in $\underline{\mathcal{H}}_\zeta$ of the function $\underline{\Omega} : \lambda \mapsto \Omega = \Psi[\mathbf{1}]$. Indeed we have

$$\begin{aligned} \|\mathbf{v}_\zeta[\pi_{\underline{\omega}_\zeta}(\mathbf{a})\Omega_{\underline{\omega}_\zeta}]\|^2 &= \langle \underline{\pi}_\zeta(\mathbf{a})\underline{\Omega}_\zeta, \underline{\pi}_\zeta(\mathbf{a})\underline{\Omega}_\zeta \rangle \\ &= \int \mathrm{d}\zeta(\lambda)\ \langle \Omega, \mathbf{a}(\lambda)^*\mathbf{a}(\lambda)\Omega \rangle \\ &= \langle \underline{\omega}_\zeta, \mathbf{a}(\lambda)^*\mathbf{a}(\lambda) \rangle \end{aligned}$$

and $\mathbf{v}_\zeta$ is a well defined isometry which intertwines the representations $\underline{\pi}_\zeta$ and $\pi_{\underline{\omega}_\zeta}$. $\square$

D The proof of Theorem 3.3

Construction of an intertwining isometry. We have shown in the previous paragraph that the Hilbert space $\underline{\mathcal{H}}_\zeta$ carries a faithful representation of the scaling algebra $\underline{\mathfrak{A}}_\zeta$. On the other hand, the Hilbert space $\mathcal{H}_\zeta$ carries a faithful representation of the algebra $\mathfrak{A}_\zeta$ which is constructed from the scaling limit $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$ of the euclidean field $(\mathfrak{B}, \beta, \eta)$. Within this paragraph we construct an isometry from $\mathcal{H}_\zeta$ to $\underline{\mathcal{H}}_\zeta$. It turns out that this particular isometry induces an algebra isomorphism between $\mathfrak{A}_\zeta$ and $\underline{\mathfrak{A}}_\zeta$.

For each $s \in \mathbb{R}_+$ and for each $\mathbf{b} \in \underline{\mathfrak{B}}(s,e)$ and for each time zero operator $\mathbf{b}_0 \in \hat{\underline{\mathfrak{B}}}(e^\perp)$ we define bounded operators $\underline{\pi}_{(s,\zeta)}(\mathbf{b})$ and $\underline{\pi}_\zeta(\mathbf{b}_0)$ on $\underline{\mathcal{H}}_\zeta$ according to

$$\begin{aligned} \underline{\pi}_{(s,\zeta)}(\mathbf{b})\mathbf{q}_\zeta[\underline{\psi}] &= \int^\oplus \mathrm{d}\zeta(\lambda)\ \pi_s(\mathbf{b}(\lambda))\underline{\psi}_\lambda \\ \underline{\pi}_\zeta(\mathbf{b}_0)\mathbf{q}_\zeta[\underline{\psi}] &= \int^\oplus \mathrm{d}\zeta(\lambda)\ \pi(\mathbf{b}_0(\lambda))\underline{\psi}_\lambda \end{aligned}$$

for each $\underline{\psi} \in \underline{\mathcal{H}}$.

Lemma D.1 : *There exists an isometry*

$$\mathbf{u}_\zeta : \mathcal{H}_\zeta \to \underline{\mathcal{H}}_\zeta$$

such that for each $s \in \mathbb{R}_+$ and for each $\mathbf{b} \in \underline{\mathfrak{B}}(s,e)$ and for each time zero operator $\mathbf{b}_0 \in \hat{\underline{\mathfrak{B}}}(e^\perp)$ the identities

$$\begin{aligned} \mathbf{u}_\zeta \pi_{(s,\zeta)}(\mathbf{b}) &= \underline{\pi}_{(s,\zeta)}(\mathbf{b})\mathbf{u}_\zeta \\ \mathbf{u}_\zeta \pi_\zeta(\mathbf{b}_0) &= \underline{\pi}_\zeta(\mathbf{b}_0)\mathbf{u}_\zeta \end{aligned}$$

are valid.

Proof. We define the operator $\mathbf{u}_\zeta$ by according to

$$\mathbf{u}_\zeta \Psi_\zeta[\mathbf{b}] := \underline{\Psi}_\zeta[\mathbf{b}]$$

where $\underline{\Psi}[\mathbf{b}] \in \underline{\mathcal{H}}$ is defined by

$$\underline{\Psi}[\mathbf{b}](\lambda) := \Psi[\mathbf{b}(\lambda)]$$

and $\underline{\Psi}_\zeta[\mathbf{b}]$ is the corresponding equivalence class in $\underline{\mathcal{H}}_\zeta$. Indeed, $\mathbf{u}_\zeta$ is a well defined isometry, since we have

$$
\begin{aligned}
\|\mathbf{u}_\zeta \Psi_\zeta[\mathbf{b}]\|^2 &:= \langle \underline{\Psi}[\mathbf{b}], \underline{\Psi}[\mathbf{b}]\rangle_\zeta \\
&= \int \mathrm{d}\zeta(\lambda)\ \langle \eta, j_e(\mathbf{b}(\lambda))\mathbf{b}(\lambda)\rangle \\
&= \langle \underline{\eta}_\zeta, j_e(\mathbf{b})\mathbf{b}\rangle \\
&= \|\Psi_\zeta[\mathbf{b}]\|^2 \ .
\end{aligned}
$$

We compute for each $s \in \mathbb{R}_+$, for each $\mathbf{b} \in \underline{\mathfrak{B}}(s,e)$ and for each $\mathbf{b}_1 \in \underline{\mathfrak{B}}(e)$:

$$
\begin{aligned}
\mathbf{u}_\zeta \pi_{(s,\zeta)}(\mathbf{b})\Psi_\zeta[\mathbf{b}_1] &= \mathbf{u}_\zeta \Psi_\zeta[\mathbf{b}\underline{\beta}_{se}\mathbf{b}_1] \\
&= \underline{\Psi}_\zeta[\mathbf{b}\underline{\beta}_{se}\mathbf{b}_1] \\
&= \underline{\pi}_{(s,\zeta)}(\mathbf{b})\underline{\Psi}_\zeta[\mathbf{b}_1] \\
&= \underline{\pi}_{(s,\zeta)}(\mathbf{b})\mathbf{u}_\zeta\Psi_\zeta[\mathbf{b}_1] \ .
\end{aligned}
$$

The identity

$$
\mathbf{u}_\zeta \pi_\zeta(\mathbf{b}_0) = \underline{\pi}_\zeta(\mathbf{b}_0)\mathbf{u}_\zeta
$$

follows from a similar computation which implies the lemma. □

Lemma D.2 : *The isometry $\mathbf{u}_\zeta$ intertwines the representations U_ζ and $\underline{U}_\zeta$: For each Poincaré transformation g the identity*

$$
\mathbf{u}_\zeta U_\zeta(g) = \underline{U}_\zeta(g)\mathbf{u}_\zeta
$$

holds true.

Proof. Let e_1 be a euclidean direction, perpendicular to e. and let $\beta_{(e,e_1)}$ be the one-parameter group of automorphisms which are given by the rotations in the e, e_1 plane. Let $\Gamma(e,r)$, $r > 0$, be the open cone which is invariant under the stabilizer subgroup of e and which has opening angle $\pi/2 - r$ with $r \in (0, \pi/2)$.

According to [13], for each τ, with $|\tau| \leq r$, there exist self adjoint operators

$$V_{(e,e_1)}(\tau) : \Psi[\mathfrak{B}(\Gamma(e,r))] \quad \rightarrow \quad \mathcal{H}$$

$$V_{(e,e_1,\zeta)}(\tau) : \Psi_\zeta[\underline{\mathfrak{B}}(\Gamma(e,r))] \quad \rightarrow \quad \mathcal{H}_\zeta$$

which are given by

$$V_{(e,e_1)}(\tau)\Psi[b] \quad := \quad \Psi[\beta_{(e,e_1,\tau)}b]$$

$$V_{(e,e_1,\zeta)}(\tau)\Psi_\zeta[\mathbf{b}] \quad := \quad \Psi_\zeta[\underline{\beta}_{(e,e_1,\tau)}\mathbf{b}] \quad .$$

We identify the hyperplane $e^\perp$ with a space like hyperplane in Minkowski space and e with the corresponding timelike direction. Let $B_{(e,e_1)}$ and $B_{(e,e_1,\zeta)}$ be the (anti-selfadjoint) generators of the Lorentz boosts in e, e_1 direction within the representation U and U_ζ respectively. Then the identities

$$V_{(e,e_1)}(\tau) \quad = \quad \exp(\mathrm{i}\tau B_{(e,e_1)})$$

$$V_{(e,e_1,\zeta)}(\tau) \quad = \quad \exp(\mathrm{i}\tau B_{(e,e_1,\zeta)})$$

are valid. For operators $\mathbf{b}_0 \in \underline{\mathfrak{B}}(e)$ and $\mathbf{b}_1 \in \underline{\mathfrak{B}}(\Gamma(e,r))$, we introduce complex functions by

$$\underline{F}_\lambda^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(z) \quad := \quad \langle\Psi[\mathbf{b}_0(\lambda)], \exp(zB_{(e,e_1)})\Psi[\mathbf{b}_1(\lambda)]\rangle$$

$$F_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(z) \quad := \quad \langle\Psi_\zeta[\mathbf{b}_0], \exp(zB_{(e,e_1,\zeta)})\Psi_\zeta[\mathbf{b}_1]\rangle \quad .$$

According to the analysis, carried out in [13, 22], the functions $\underline{F}_\lambda^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$, $\lambda > 0$, and $F_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$ are holomorphic in the open strip $\mathbb{R} + \mathrm{i}(-r, r)$.

Furthermore, let H_e and $H_{e,\zeta}$ be the positive generators (Hamilton operators) of the semi group of contractions V_e and $V_{(e,\zeta)}$ respectively, given by

$$V_e(\tau)\Psi[b] \quad := \quad \Psi[\beta_{\tau e}b]$$

$$V_{(e,\zeta)}(\tau)\Psi_\zeta[\mathbf{b}] \quad := \quad \Psi_\zeta[\underline{\beta}_{\tau e}\mathbf{b}]$$

for each $\tau > 0$ and for each $b \in \mathfrak{B}(e)$ and for each $\mathbf{b} \in \underline{\mathfrak{B}}(e)$. For $\mathbf{b}_0, \mathbf{b}_1 \in \underline{\mathfrak{B}}(e)$ we introduce again complex functions

$$\underline{F}_\lambda^{[e|\mathbf{b}_0,\mathbf{b}_1]}(z) \quad := \quad \langle\Psi[\mathbf{b}_0(\lambda)], \exp(\lambda z H_e)\Psi[\mathbf{b}_1(\lambda)]\rangle$$

$$F_\zeta^{[e|\mathbf{b}_0,\mathbf{b}_1]}(z) \quad := \quad \langle\Psi_\zeta[\mathbf{b}_0], \exp(zH_{(e,\zeta)})\Psi_\zeta[\mathbf{b}_1]\rangle \quad .$$

which are holomorphic in the upper half plane $\mathbb{R} + \mathrm{i}\mathbb{R}_+$.

The lemma follows mainly from the subsequent statements, which are proven in the next section Appendix E:

Sublemma D.3 : *For each limit functional* $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$*, for each* $\mathbf{b}_0 \in \underline{\mathfrak{B}}(e)$*, and for each* $\mathbf{b}_1 \in \underline{\mathfrak{B}}(\Gamma(e,r))$*:*

(1) The prescription

$$\underline{F}_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]} : z \quad \mapsto \quad \int \mathrm{d}\zeta(\lambda) \ \underline{F}_\lambda^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(z)$$

yields a well defined function which is holomorphic in the strip $\mathbb{R} + \mathrm{i}(-r,r)$*.*

(2) The prescription

$$\underline{F}_\zeta^{[e|\mathbf{b}_0,\mathbf{b}_1]} : z \quad \mapsto \quad \int \mathrm{d}\zeta(\lambda) \ \underline{F}_\lambda^{[e|\mathbf{b}_0,\mathbf{b}_1]}(z)$$

yields a well defined function which is holomorphic in the upper half plane $\mathbb{R} + \mathrm{i}\mathbb{R}_+$*.*

Sublemma D.4 : *For each limit functional* $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$*, for each* $\mathbf{b}_0 \in \underline{\mathfrak{B}}(e)$*, and for each* $\mathbf{b}_1 \in \underline{\mathfrak{B}}(\Gamma(e,r))$ *the identities*

$$\underline{F}_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]} \quad = \quad F_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$$

$$\underline{F}_\zeta^{[e|\mathbf{b}_0,\mathbf{b}_1]} \quad = \quad F_\zeta^{[e|\mathbf{b}_0,\mathbf{b}_1]}$$

holds true.

Let $e_1 \perp e$ and let g be the Lorentz transformation such that

$$U(g) \quad = \quad \exp(tB_{(e,e_1)}) \quad .$$

According to Sublemma D.4 we compute

$$\begin{aligned} \langle \Psi_\zeta[\mathbf{b}_0], \mathbf{u}_\zeta^* \underline{U}_\zeta(g) \underline{\Psi}_\zeta[\mathbf{b}_1] \rangle \quad &= \quad \underline{F}_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(t) \\ &= \quad F_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(t) \\ &= \quad \langle \Psi_\zeta[\mathbf{b}_0], U_\zeta(g) \Psi_\zeta[\mathbf{b}_1] \rangle \end{aligned}$$

and since $\Psi_\zeta[\underline{\mathfrak{B}}(e)]$ is dense in $\mathcal{H}_\zeta$, we conclude that

$$\underline{U}_\zeta(g) \underline{\Psi}_\zeta[\mathbf{b}_1] \quad = \quad \mathbf{u}_\zeta U_\zeta(g) \Psi_\zeta[\mathbf{b}_1]$$

which implies

$$\mathbf{u}_\zeta U_\zeta(g) = \underline{U}_\zeta(g)\mathbf{u}_\zeta$$

for each Lorentz boost g. Analogously we conclude for the time like translations in e direction

$$\begin{aligned}\langle \Psi_\zeta[\mathbf{b}_0], \mathbf{u}_\zeta^* \underline{U}_\zeta(te)\underline{\Psi}_\zeta[\mathbf{b}_1]\rangle &= \underline{F}_\zeta^{[e|\mathbf{b}_0,\mathbf{b}_1]}(t) \\ &= F_\zeta^{[e|\mathbf{b}_0,\mathbf{b}_1]}(t) \\ &= \langle \Psi_\zeta[\mathbf{b}_0], U_\zeta(te)\Psi_\zeta[\mathbf{b}_1]\rangle\end{aligned}$$

and thus

$$\mathbf{u}_\zeta U_\zeta(te) = \underline{U}_\zeta(te)\mathbf{u}_\zeta$$

for each $t \in \mathbb{R}$.

Now, let g be an element of the stabilizer subgroup of the hyperplane $e^\perp$, then we compute for each $\mathbf{b} \in \underline{\mathfrak{B}}(e)$:

$$\begin{aligned}\mathbf{u}_\zeta U_\zeta(g)\Psi_\zeta[\mathbf{b}] &= \mathbf{u}_\zeta \Psi_\zeta[\underline{\beta}_g \mathbf{b}] \\ &= \underline{\Psi}_\zeta[\underline{\beta}_g \mathbf{b}] \\ &= \underline{U}_\zeta(g)\underline{\Psi}_\zeta[\mathbf{b}] \quad .\end{aligned}$$

Therefore the identity

$$\mathbf{u}_\zeta U_\zeta(g) = \underline{U}_\zeta(g)\mathbf{u}_\zeta$$

is valid for each Poncaré transformation g which completes the proof.
□

Proof of Theorem 3.3. By Lemma C.1 the quantum field $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$ may be identified with the scaling algebra $(\underline{\mathfrak{A}}, \underline{\alpha})$ in the vacuum representation $\underline{\pi}_\zeta$.

Recall, that the scaling algebra $\underline{\mathfrak{A}}$ can be introduced in terms of the time-zero algebras $\hat{\underline{\mathfrak{B}}}(\mathscr{V})$, where $\mathscr{V}$ is a bounded and convex region in the time slice $e^\perp$. For a double cone $\mathscr{O}$, the local scaling algebra $\underline{\mathfrak{A}}(\mathcal{O})$ is generated by all functions

$$\underline{\Pi}[f, \mathbf{b}] : \lambda \quad \mapsto \quad \int \mathrm{d}g \ f(g) \ U_\lambda(g)\pi(\mathbf{b}(\lambda))U_\lambda(g)^*$$

where $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$ is a smooth function on the Poincaré group with compact support such that $g\mathscr{V} \subset \mathscr{O}$ for each g in the support of f and $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}(\mathscr{V})$. Applying the representation $\underline{\pi}_\zeta$ to $\underline{\Pi}[f,\mathbf{b}]$ yields

$$\underline{\Pi}_\zeta[f,\mathbf{b}] \quad := \quad \underline{\pi}_\zeta(\underline{\Pi}[f,\mathbf{b}]) \quad = \quad \int \mathrm{d}g \ f(g) \ \underline{U}_\zeta(g)\underline{\pi}_\zeta(\mathbf{b})\underline{U}_\zeta(g)^* \quad .$$

On the other hand, for a double cone $\mathscr{O}$, the local algebra $\mathfrak{A}_\zeta(\mathscr{O})$ of the quantum field $(\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$ which can be constructed from the euclidean field $(\underline{\mathfrak{B}}, \underline{\beta}, \underline{\eta}_\zeta)$, is generated by operators of the form

$$\Pi_\zeta[f,\mathbf{b}] \quad = \quad \int \mathrm{d}g \ f(g) \ U_\zeta(g)\pi_\zeta(\mathbf{b})U_\zeta(g)^*$$

with a time-zero operator $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}(\mathscr{V})$ and a smooth function $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$, such that $g\mathscr{V} \subset \mathscr{O}$ for each g in the support of f.

According to the intertwining properties of the isometry $\mathbf{u}_\zeta$ we conclude form Lemma D.1 and from Lemma D.2 that the identity

$$\begin{aligned} & \mathbf{u}_\zeta \left[\int \mathrm{d}g \ f(g) \ U_\zeta(g)\pi_{(s,\zeta)}(\mathbf{b})U_\zeta(g)^* \right] \mathbf{u}_\zeta^* \\ = & \int \mathrm{d}g \ f(g) \ \mathbf{u}_\zeta U_\zeta(g)\pi_{(s,\zeta)}(\mathbf{b})U_\zeta(g)^*\mathbf{u}_\zeta^* \\ = & \int \mathrm{d}g \ f(g) \ \underline{U}_\zeta(g)\underline{\pi}_{(s,\zeta)}(\mathbf{b})\underline{U}_\zeta(g)^* \end{aligned}$$

is valid for each $s \in \mathbb{R}_+$. This implies

$$\mathbf{u}_\zeta \Pi_\zeta[f,\mathbf{b}]\mathbf{u}_\zeta^* \quad = \quad \underline{\Pi}_\zeta[f,\mathbf{b}]$$

for each smooth function $f \in \mathscr{C}_0^\infty(\mathrm{P}_+^\uparrow)$ and for each time-zero operator $\mathbf{b} \in \hat{\underline{\mathfrak{B}}}(e^\perp)$. Therefore, we get

$$\mathbf{u}_\zeta \mathfrak{A}_\zeta(\mathcal{O})\mathbf{u}_\zeta^* \quad := \quad \underline{\pi}_\zeta(\underline{\mathfrak{A}}(\mathcal{O})) \quad = \quad \underline{\mathfrak{A}}_\zeta(\mathcal{O})$$

and the map

$$\iota_\zeta : \mathfrak{A}_\zeta \to \underline{\mathfrak{A}}_\zeta \quad ; \quad \mathbf{a} \mapsto \mathbf{u}_\zeta \mathbf{a} \mathbf{u}_\zeta^*$$

yields an isomorphism of the quantum fields $(\mathfrak{A}_\zeta, \alpha_\zeta, \omega_\zeta)$ and $(\underline{\mathfrak{A}}_\zeta, \underline{\alpha}_\zeta, \underline{\omega}_\zeta)$ since

$$\iota_\zeta \circ \alpha_\zeta \quad = \quad \underline{\alpha}_\zeta \circ \iota_\zeta$$

is valid according to Lemma D.2 and

$$\omega_\zeta \quad = \quad \underline{\omega}_\zeta \circ \iota_\zeta$$

holds true, which is a consequence of the fact that $\mathbf{u}_\zeta \Omega_\zeta = \underline{\Omega}_\zeta$, where $\underline{\Omega}_\zeta$ is the equivalence class in $\underline{\mathcal{H}}_\zeta$ of the constant function $\lambda \mapsto \Omega$. $\square$

E Remarks on holomorphic functions

The statements of Sublemma D.3 and Sublemma D.4 can directly be obtained form some general statement on holomorphic functions, which we discuss in the subsequent.

Lemma E.1 : *Let $I \subset \mathbb{R}$ be an open connected subset and let $f_\lambda \in \mathcal{O}(\mathbb{R} + \mathrm{i}I)$, $\lambda \in \mathbb{R}_+$, be a family of functions which are holomorphic in $\mathbb{R} + \mathrm{i}I$. If there exists a constant $K > 0$ such that the bound*

$$|f_\lambda(z)| \quad \leq \quad K$$

holds true for each $\lambda \in \mathbb{R}_+$ and for each $z \in \mathbb{R} + \mathrm{i}I$, then for each limit functional $\zeta \in \mathfrak{S}[\mathrm{C}(\mathbb{R}_+)]$ there exists a function $f_\zeta \in \mathcal{O}(\mathbb{R} + \mathrm{i}I)$, holomorphic in $\mathbb{R} + \mathrm{i}I$, which is uniquely determined by the prescription

$$f_\zeta(z) \quad = \quad \int \mathrm{d}\zeta(\lambda)\ f_\lambda(z) \quad .$$

Proof. The lemma is nothing else but Montel's Theorem expressed in terms of limit functionals. According to our assumption the family of holomorphic functions $f_\lambda \in \mathcal{O}(\mathbb{R} + \mathrm{i}I)$, $\lambda \in \mathbb{R}_+$, is uniformly bounded by a constant K, i.e.

$$|f_\lambda(z)| \quad \leq \quad K$$

for each $\lambda \in \mathbb{R}_+$ and for each $z \in \mathbb{R} + \mathrm{i}I$. Let $r > 0$ and let $D_r(z)$ be the closed disc in $\mathbb{C}$ with radius r and center z. For $z \in \mathbb{R} + \mathrm{i}I$ we choose $r > 0$ with $D_r(z) \subset \mathbb{R} + \mathrm{i}I$. For $z_1, z_2 \in D_r(z)$ we immediately get the estimate

$$|f_\lambda(z_1) - f_\lambda(z_2)| \quad = \quad \left| \int_{z_1}^{z_2} \mathrm{d}z f_\lambda'(z) \right| \quad \leq \quad |z_1 - z_2| 4r^{-1} K$$

uniformly in λ. This implies, since ζ is a positive functional,

$$|f_\zeta(z_1) - f_\zeta(z_2)| \quad \leq \quad |z_1 - z_2| 4r^{-1} K$$

for each $z_1, z_2 \in D_r(z)$. Now, the function

$$f_\zeta : z \quad \mapsto \quad \int \mathrm{d}\zeta(\lambda)\ f_\lambda(z)$$

is integrable with respect to the natural measure on the circle $\partial D_r(z)$ since it is continuous in $\mathbb{R} + \mathrm{i}I$ (in particular uniformly continuous in $D_r(z)$ for each $z \in \mathbb{R} + \mathrm{i}I$). On the other hand, the function

$$f_{\partial D_r(z)} : \lambda \mapsto \int_{\partial D_r(z)} \mathrm{d}z'\ f_\lambda(z')$$

is uniformly bounded in λ, i.e. $f_{\partial D_r(z)}$ is contained in $\mathscr{F}_{\mathrm{b}}(\mathbb{R}_+)$, and hence $f_{\partial D_r(z)}$ is measurable with respect to ζ. Thus Fubini's theorem can be applied which states that the integration over $\partial D_r(z)$ and the integration with respect to the measure ζ can be exchanged. This gives

$$\begin{aligned}\int_{\partial D_r(z)} \mathrm{d}z' \ f_\zeta(z') &= \int_{\partial D_r(z)} \mathrm{d}z' \int \mathrm{d}\zeta(\lambda) \ f_\lambda(z') \\ &= \int \mathrm{d}\zeta(\lambda) \int_{\partial D_r(z)} \mathrm{d}z' \ f_\lambda(z') \\ &= 0\end{aligned}$$

which implies that f_ζ is holomorphic in z. $\square$

Proof of Sublemma D.3. In order to prove Sublemma D.3 we show that the functions

$$\begin{aligned}\underline{F}_\lambda^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(z) &= \langle \Psi[\mathbf{b}_0(\lambda)], \exp(zB_{(e,e_1)})\Psi[\mathbf{b}_1(\lambda)]\rangle \\ F_\lambda^{[e|\mathbf{b}_0,\mathbf{b}_1]}(z) &= \langle \Psi_\zeta[\mathbf{b}_0], \exp(\lambda z H_e)\Psi_\zeta[\mathbf{b}_1]\rangle\end{aligned}$$

are uniformly bounded. Since the operators $\exp(zB_{(e,e_1)})$ and $\exp(\lambda z H_e)$ are unitary for real z and since the norm of the vector $\Psi[b] \in \mathcal{H}$, $b \in \mathfrak{B}(e)$, is bounded by the operator norm $||b||$ we conclude the estimate

$$\begin{aligned}|\underline{F}_\lambda^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(z)| &\leq ||\mathbf{b}_0|| \ ||\mathbf{b}_1|| \\ |\underline{F}_\lambda^{[e|\mathbf{b}_0,\mathbf{b}_1]}(z)| &\leq ||\mathbf{b}_0|| \ ||\mathbf{b}_1||\end{aligned}$$

for each z in the corresponding region of holomorphy and for each $\lambda \in \mathbb{R}_+$. Thus Lemma E.1 can be applied, which proves Sublemma D.3. $\square$

A further well known fact concerning holomorphic function is the following:

Lemma E.2 : *$f, \underline{f} \in \mathcal{O}(\mathbb{R} + \mathrm{i}I)$, be two functions which are holomorphic in $\mathbb{R} + \mathrm{i}I$, where I is connected. If f and $\underline{f}$ coincide within the imaginary points $\mathrm{i}I$, then $f = \underline{f}$.*

Proof of Sublemma D.4. For each $\mathbf{b}_1 \in \underline{\mathfrak{B}}(\Gamma(e,r))$, the functions $\underline{F}_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$ and $F_\zeta^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$ are holomorphic in the open strip $\mathbb{R} +$

$\mathrm{i}(-r,r)$ and one easily computes for each $\tau \in (-r,r)$

$$\begin{aligned}
\underline{F}_{\zeta}^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(\mathrm{i}\tau) &= \int \mathrm{d}\zeta(\lambda)\ \langle \Psi[\mathbf{b}_0(\lambda)], \exp(\mathrm{i}\tau B_{(e,e_1)})\Psi[\mathbf{b}_1(\lambda)]\rangle \\
&= \int \mathrm{d}\zeta(\lambda)\ \langle \Psi[\mathbf{b}_0(\lambda)], \Psi[\underline{\beta}_{(e,e_1,\tau)}\mathbf{b}_1(\lambda)]\rangle \\
&= \int \mathrm{d}\zeta(\lambda)\ \langle \eta, j_e(\mathbf{b}_0(\lambda))\underline{\beta}_{(e,e_1,\tau)}\mathbf{b}_1(\lambda)]\rangle \\
&= \langle \eta_\zeta, j_e(\mathbf{b}_0)\underline{\beta}_{(e,e_1,\tau)}\mathbf{b}_1\rangle \\
&= \langle \Psi_\zeta[\mathbf{b}_0], \exp(\mathrm{i}\tau B_{(e,e_1,\zeta)})\Psi_\zeta[\mathbf{b}_1]\rangle]\rangle \\
&= F_{\zeta}^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}(\mathrm{i}\tau)
\end{aligned}$$

which implies that $F_{\zeta}^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$ and $\underline{F}_{\zeta}^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$ coincide in the imaginary points and by Lemma E.2 it follows that $F_{\zeta}^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]} = \underline{F}_{\zeta}^{[e,e_1|\mathbf{b}_0,\mathbf{b}_1]}$. Analogously one proves that $F_{\zeta}^{[e|\mathbf{b}_0,\mathbf{b}_1]} = \underline{F}_{\zeta}^{[e|\mathbf{b}_0,\mathbf{b}_1]}$ holds true also. $\Box$

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